Calculating the Probability for Neutrino Oscillations

Student Lecture Series for MiniBooNE

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1   Neutrino Oscillations

The weak eigenstates that we normally observe (ν_μ, ν_e) can oscillate between each other if they are composed of an add-mixture of mass eigenstates (ν_1, ν_2). If the weak eigenstates are rotated by an angle θ with respect to the mass eigenstates (Fig. 1), then a matrix equation can be written that relates the weak eigenstates to the mass eigenstates (see below). For example, using the matrix equation below, the ν_e state can be written as $|\nu_e\rangle = \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle$, where θ is called the *mixing angle*.

$$
\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}
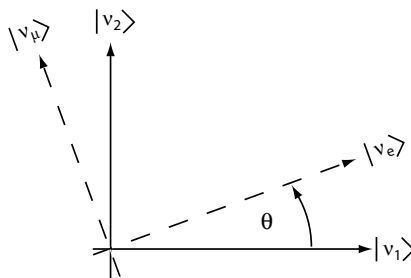$$

Figure 1: The weak eigenstates are rotated by an angle θ with respect to the mass eigenstates (ν_1 and ν_2) to allow mixing (i.e., oscillations) between the ν_μ and ν_e.

The mass eigenstates (ν_1, ν_2) have masses m_1 and m_2 and both have momentum p.

2   Look at the time evolution of the ν_μ state

$$|\nu_\mu(t=0)\rangle \;=\; |\nu_\mu\rangle \;=\; -\sin\theta\,|\nu_1\rangle \;+\; \cos\theta\,|\nu_2\rangle \tag{1}$$

$$|\nu_\mu(t)\rangle \;=\; -\sin\theta\,|\nu_1\rangle\,e^{-i\frac{E_1 t}{\hbar}} \;+\; \cos\theta\,|\nu_2\rangle\,e^{-i\frac{E_2 t}{\hbar}} \tag{2}$$

where $E_1 = \sqrt{p^2 c^2 + m_1^2 c^4}$ and $E_2 = \sqrt{p^2 c^2 + m_2^2 c^4}$ and $p_1 = p_2$.

3   Some Approximations

Let $\hbar = c = 1$.

Then $E_1 = \sqrt{p^2 + m_1^2}$ and $E_2 = \sqrt{p^2 + m_2^2}$

Also, the neutrinos are assumed to be relativistic:

$$\gamma = \frac{E}{m_o c^2} = \frac{\sqrt{p^2 c^2 + m_o^2 c^4}}{m_o c^2} \gg 1 \tag{3}$$

$$\text{then} \quad p \gg m_o \tag{4}$$

$$E \;=\; \sqrt{p^2 + m_o^2} \;=\; p\sqrt{1 + m_o^2/p^2} \;\simeq\; p + \frac{1}{2}\frac{m_o^2}{p} \tag{5}$$

where we use the binomial expansion: $(1+x)^n \simeq 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots$ and keep just the first two terms.

The energy of the two mass eigenstates can be written approximately as:

$$E_1 \simeq p + \frac{1}{2}\frac{m_1^2}{p} \quad \text{and likewise} \quad E_2 \simeq p + \frac{1}{2}\frac{m_2^2}{p} \;. \tag{6}$$

4 How does the ν_μ propagate in time?

$$|\nu_\mu(t)\rangle = -\sin\theta \; |\nu_1\rangle \; e^{-i\left(p + \frac{1}{2}\frac{m_1^2}{p}\right)t} + \cos\theta \; |\nu_2\rangle \; e^{-i\left(p + \frac{1}{2}\frac{m_2^2}{p}\right)t} \tag{7}$$

$$|\nu_\mu(t)\rangle = e^{-i\left(p + \frac{1}{2}\frac{m_1^2}{p}\right)t}\left(-\sin\theta \; |\nu_1\rangle \; + \cos\theta \; |\nu_2\rangle \; e^{+i\left(\frac{1}{2}\frac{m_1^2 - m_2^2}{p}\right)t}\right) \tag{8}$$

Now, for some definitions and substitutions:

$$\Delta m^2 = m_1^2 - m_2^2 \quad \text{and} \quad t = \frac{x}{c} = x \quad \text{and} \quad e^{-iz} = e^{-i\left(p + \frac{1}{2}\frac{m_1^2}{p}\right)t} \tag{9}$$

Now, we have:

$$|\nu_\mu(t)\rangle = e^{-iz}\left(-\sin\theta \; |\nu_1\rangle \; + \cos\theta \; |\nu_2\rangle \; e^{+i\left(\frac{1}{2}\frac{\Delta m^2}{p}\right)x}\right) \tag{10}$$

5 What is the probability for $\nu_\mu \to \nu_e$?

To calculate the probability for a "pure" ν_μ state to oscillate into a ν_e state, we must square the quantum mechanical amplitude that describes this transition.

$$P\left(\nu_\mu \to \nu_e\right) \; = \; |\langle\nu_e|\nu_\mu(t)\rangle|^2 \tag{11}$$

Recall from section (1) that

$$\langle\nu_e| \; = \; \cos\theta \; \langle\nu_1| \; + \; \sin\theta \; \langle\nu_2| \tag{12}$$

So, now we can write the amplitude as:

$$\langle\nu_e|\nu_\mu(t)\rangle = e^{-iz}\left(-\sin\theta\cos\theta + \sin\theta\cos\theta \; e^{\frac{i}{2}\frac{\Delta m^2}{p}x}\right) \tag{13}$$

where we use the relationship $\langle\nu_i|\nu_j\rangle = \delta_{ij}$.

Taking the absolute value squared, we find that:

$$P\left(\nu_\mu \to \nu_e\right) \;=\; |\langle \nu_e | \nu_\mu(t) \rangle|^2 \;=$$

$$= \; e^{+iz} e^{-iz} \sin^2\theta \cos^2\theta \left(-1 + e^{\frac{i}{2}\frac{\Delta m^2}{p}x}\right)\left(-1 + e^{-\frac{i}{2}\frac{\Delta m^2}{p}x}\right)$$

Since the neutrino is relativistic, we can also make the substitution: $p = E_\nu$, and likewise, we will make the substitution $x = L$.

$$P\left(\nu_\mu \to \nu_e\right) \;=\; \frac{1}{2}\sin^2 2\theta \left(1 - \cos\left(\frac{\Delta m^2}{2}\frac{L}{E_\nu}\right)\right) \;. \qquad (14)$$

Using the trigonometric relation $(1 - \cos 2\theta)/2 = \sin^2\theta$, we can write the above equation as

$$P\left(\nu_\mu \to \nu_e\right) \;=\; \sin^2 2\theta \; \sin^2\left(\frac{\Delta m^2}{4}\frac{L}{E_\nu}\right) \;. \qquad (15)$$

Now, we can write the argument of the second $\sin^2$ term above so it's dimensionless by introducing the appropriate number of $\hbar$'s and c's.

$$\left(\frac{\Delta m^2}{4}\frac{L}{E_\nu}\right) \;\Rightarrow\; \left(\frac{\Delta m^2 c^4}{4\,\hbar c}\frac{L}{E_\nu}\right)$$

Let's write the above quantity in units that are convenient for an experimental physicist. We would like the variables in the above equation to have the following units: $\Delta m^2 c^4 (\mathrm{eV}^2)$, L (meters), and E_ν (MeV). If we substitute $\hbar c$ with 197 eV·nm, we can write the quantity in parenthesis as

$$\left(\frac{\Delta m^2 c^4}{4\,\hbar c}\frac{L}{E_\nu}\right) \;\Rightarrow\; \left(\frac{\Delta m^2 c^4}{4\times\,197\,\mathrm{eV}\cdot\mathrm{nm}}\frac{L}{E_\nu}\right)\left(\frac{10^{-6}\mathrm{MeV}/\mathrm{eV}}{10^{-9}\mathrm{m}/\mathrm{nm}}\right) = \left(1.27\,\Delta m^2\,\frac{L}{E_\nu}\right)$$

Finally, we can write Eq. 15 in its more familiar form:

$$P_{\nu_\mu \to \nu_e}(L, E) \;=\; \sin^2 2\theta \; \sin^2\left(1.27\,\Delta m^2 \frac{L}{E_\nu}\right) \;. \qquad (16)$$

If neutrino oscillations occur, the mixing probability $(\sin^2 2\theta)$ and the mass difference (Δm^2) are determined by nature. Physicists can probe different regions of Δm^2 by adjusting the distance between the neutrino source and the detector (L) as well as the neutrino energy
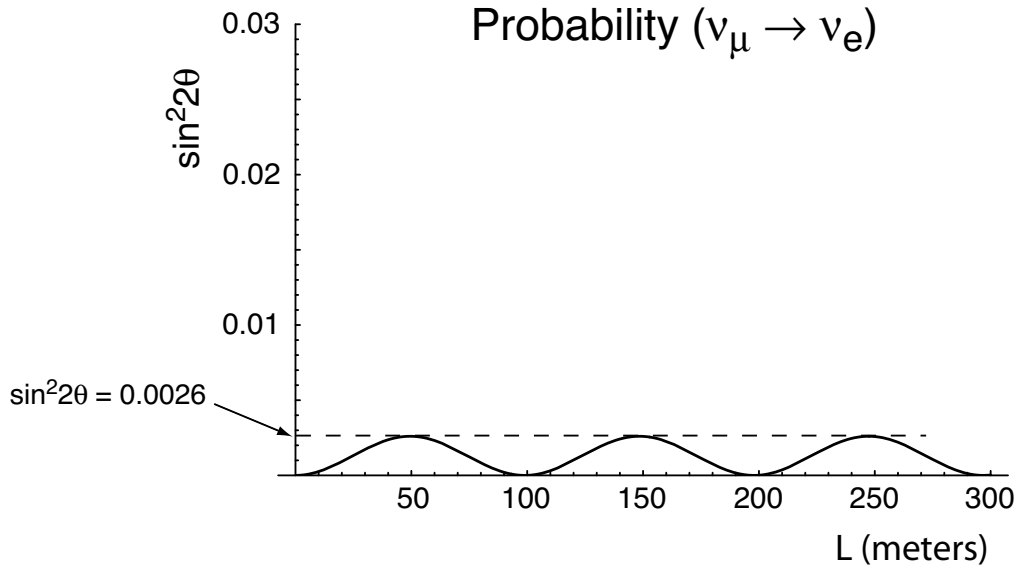
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Figure 2: Neutrino oscillations (ν_e appearance) as a function of length for a monoenergetic beam of ν_μ's. There is a maximum probability to observe ν_e interactions at 50, 150, and 250 meters.

E_ν. If a mono-energetic beam of neutrinos is produced (e.g., 40 MeV) and the mixing parameters suggested by LSND are correct (e.g., $\sin^2 2\theta = 0.0026$ and $\Delta m^2 \simeq 1$ eV), then it's possible to plot the oscillation of $\nu_\mu \to \nu_e$ using Eq. 16. A small fraction of the initial ν_μ beam ($\sin^2 2\theta$) appears as ν_e's as shown in Fig. 2.

Maximal mixing occurs if $\sin^2 2\theta$ is $\simeq 1$ (i.e., $\theta = 45^o$). In the case of atmospheric neutrinos, it is suspected that maximal mixing occurs. If this were the case, the peaks in Fig. 2 would oscillate between 0 and 1.0 on the vertical axis–assuming all the atmospheric neutrinos were monoenergetic (which is not the case in real life).

The L/E term is the quantity of interest when exploring different mass regions. In the LSND experiment, L was about 30 meters and E was about 30 MeV giving an L/E of ~ 1. If MiniBooE is going to explore the same Δm^2 and $\sin^2 \theta$ region, then its L/E must be similar to the LSND value. In the case of MiniBooNE, the neutrinos travel about 500 m and have energies on the order of 500 MeV. So, roughly speaking, the MiniBooNE experiment is designed to explore the same Δm^2 region as LSND, but with higher sensitivity (i.e., down to lower values of $\sin^2 2\theta$).

6   Real Neutrino Beams

In the previous section, we discussed where the ν_e appearance occurs as a function of distance when we have a monoenergetic neutrino beam. In reality, most neutrino sources have

a range of energies which will tend to "wash out" the distribution shown in Fig. 2.

To investigate the changes to the amplitude and oscillation length, we explore two possibilities for neutrino energy distributions–a *narrow band* distribution, and a *wide band* distribution. In the first case, we assume that the neutrino momentum is well-defined, that is, $\Delta p/p < 5\%$. The neutrino energy distribution is shown on the left-hand side of Fig. 3 while the resulting oscillation probability is shown as a function of length on the right hand side. The oscillation as a function of length $P_{\nu_\mu \to \nu_e}(L)$ is calculated by convoluting the energy distribution $f(E)$ on the left side of Fig. 3 with the oscillation probability $P_{\nu_\mu \to \nu_e}(L, E)$.

$$P_{\nu_\mu \to \nu_e}(L) = \int_0^\infty P_{\nu_\mu \to \nu_e}(L, E)\, f(E)\, dE = \int_0^\infty \sin^2 2\theta\, \sin^2\left(1.27\, \frac{\Delta m^2\, L}{E}\right) f(E)\, dE$$
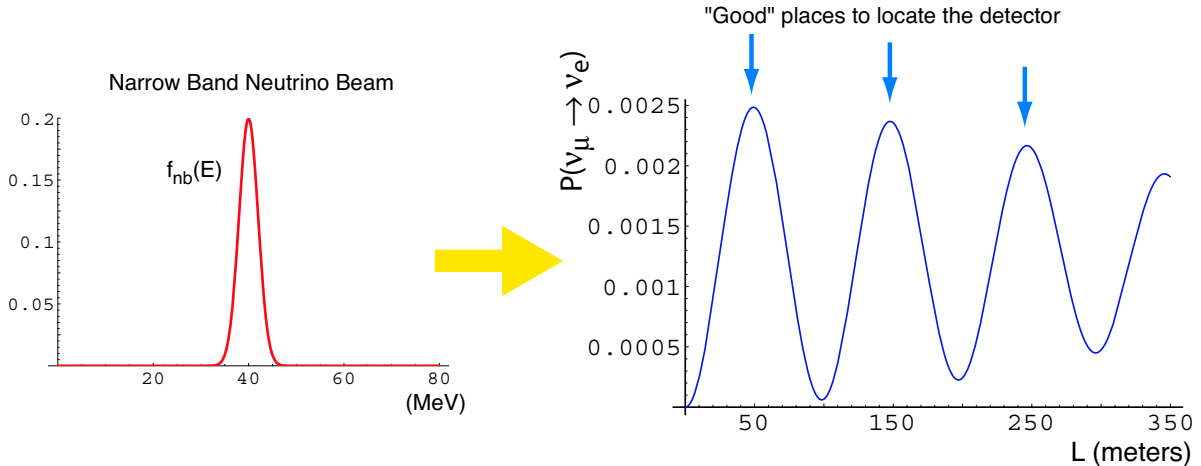

Figure 3: Neutrino oscillations (ν_e appearance) as a function of length for a monoenergetic beam of ν_μ's. There is a maximum probability to observe ν_e interactions at 50, 150, and 250 meters. The convolution integral was done using *Mathematica 4.0*

Next we investigate the oscillation probability when using a *wide band* neutrino source, $f_{wb}(E)$. A wide band neutrino beam ($\bar{\nu}_\mu$) is shown in Fig. 4 resulting from muon decays. The energy distribution is described by the following function:

$$\frac{dN}{dE} = E^2 \left(1 - \frac{2E}{3E_{max}}\right)$$

where E_{max} is 52.8 MeV for muon decay. This function is shown on the left-hand side of Fig. 4. Once again, convoluting this this energy distribution with $P_{\nu_\mu \to \nu_e}(L, E)$, we obtain the oscillating function seen on the right-hand side of Fig. 4. Notice that both the amplitude and the peaks of the oscillation ($P_{\nu_\mu \to \nu_e}(L)$) are shifted.

What about Mini-BooNE? As a homework assignment, one can use the energy distribution of ν_μ's produced by π^+ decays-in-flight and convolute this distribution to obtain the
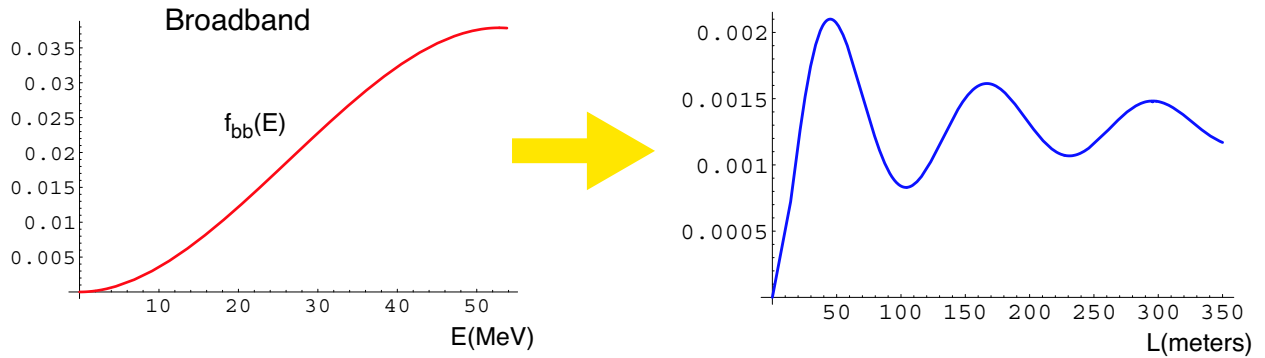
Figure 4: Neutrino oscillations (ν_e appearance) as a function of length for a wide band beam of ν_μ's. Note the pronounced oscillations from Fig. 3 are diminished due to the broadband energy distribution of neutrinos produced by decay-at-rest muons. Furthermore, the maxima are pushed to 40, 160, and 290 meters.

oscillation function $P_{\nu_\mu \to \nu_e}(L)$. This would give a first order calculation of the oscillation probability one might observe from the Mini-BooNE experiment, assuming the Δm^2 and $\sin^2 2\theta$ solutions from the LSND experiment are correct. A more precise calculation of the oscillation probability would require a full Monte-Carlo simulation of the production of neutrinos along with the geometrical and detector efficiencies of the Mini-BooNE experiment. This full-scale Monte Carlo simulation is underway.