Chapter 3
Motion in Two or Three Dimensions

1 Position and Velocity Vectors

In this chapter we investigate the simultaneous motion along the x and y directions where $a_x = 0$ and $a_y = g$. As we will see later in this chapter, that once the equations of motion are known (i.e., $x(t)$, $y(t)$, and $z(t)$), then the subsequent position $\vec{r}(t)$, velocity $\vec{v}(t)$, and acceleration $\vec{a}(t)$ can be calculated for all future times.

The position vector describes the motion of a particle moving through space. The tail of the vector is at the origin (0, 0, 0), and the head of the vector is located at the instantaneous position of the particle (x, y, z).

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

Figure 3.1 The position vector $\vec{r}$

The average velocity vector (in three dimensions) is similar to the definition we had in the previous chapter:

$$\vec{v}_{\text{av}} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1} = \frac{\Delta\vec{r}}{\Delta t} \qquad \text{(average velocity vector)}$$

where $\Delta\vec{r} = (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath} + (z_2 - z_1)\hat{k}$.

Figure 3.2 The velocity vector $\vec{v}$

Likewise, the instantaneous velocity is calculated by taking the following limit:

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} \qquad \text{(instantaneous velocity vector)}$$

$$\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k} \qquad \text{(the instantaneous velocity)}$$

where

$$v_x = \frac{dx}{dt} \qquad v_y = \frac{dy}{dt} \quad \text{and} \quad v_z = \frac{dz}{dt}$$

The instantaneous *speed* is just:

$$v = |\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

In two dimensions, where the motion is restricted to the x-y plane (e.g., projectile motion), the instantaneous *speed* becomes:

$$v = |\vec{v}| = \sqrt{v_x^2 + v_y^2}$$

where the angle of the velocity vector ($\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath}$) with respect to the x-axis is

$$\alpha = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

Figure 3.4 Velocity components in the x-y plane

Ex. 1 A squirrel has x- and y- coordinates (1.1 m, 3.4 m) at time t_1=0 and coordinates (5.3 m, -0.5 m) at time t_2=3.0 s. For this time interval, find a) the components of the average velocity; b) the magnitude and direction of the average velocity.

2 The Acceleration Vector

In this section we construct the average and instantaneous acceleration vectors as a consequence of the change in the velocity vector $\vec{v}$. The velocity vector can change as a result of a *change in magnitude*, or a *change in direction*, or both.

Figure 3.6 Constructing $\Delta\vec{v}$ and obtaining $\vec{a}_{\text{av}}$

$$\vec{a}_{\text{av}} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1} = \frac{\Delta\vec{v}}{\Delta t} \qquad \text{(average acceleration vector)}$$

Likewise, we can construct the instantaneous acceleration vector $\vec{a}$ by taking the limit

$$\vec{a} = \lim_{\Delta t \to 0}\frac{\Delta\vec{v}}{\Delta t} = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}$$

where

$$a_x = \frac{dv_x}{dt} \qquad a_y = \frac{dv_y}{dt} \quad \text{and} \quad a_z = \frac{dv_z}{dt}$$

Figure 3.10 Resolving the acceleration into $a_\parallel$ and $a_\perp$

Figure 3.11 When $\vec{a}$ is *parallel* and *perpendicular* to $\vec{v}$

Figure 3.12 The velocity and acceleration vectors for a particle moving on a *curved* path

Ex. 5 A jet plane is flying at a constant altitude. At time t_1=0 it has components of velocity v_x=90 m/s, v_y=110 m/s. At time t_2=30.0 s the components are v_x=-170 m/s, v_y=40 m/s. a) Sketch the velocity vectors at t_1 and t_2. How do these two vectors differ? For this time interval calculate b) the components of the average acceleration; c) the magnitude and direction of the average acceleration.

3 Projectile Motion

A **projectile** is any body that is given an initial velocity and then follows a path determined entirely by the effects of gravitational acceleration and air resistance. The path followed by a projectile is called its **trajectory**. If we set $a_x = 0$ and $a_y = -g$, then we have a situation where we have projectile motion with no air resistance.

In the x direction

$$v_x = v_{ox} \qquad \text{and} \qquad x = x_o + v_{ox}t$$

In the y direction

$$v_y = v_{oy} - gt \qquad \text{and} \qquad y = y_o + v_{oy}t - \frac{1}{2}gt^2$$

Figure 3.17 The velocity and acceleration vectors for a particle moving in *projectile* motion

If a projectile is launched at an angle α_o with respect to the x-axis, then

$$v_{ox} = v_o \cos\alpha_o \qquad \text{and} \qquad v_{oy} = v_o \sin\alpha_o$$

If we set $x_o = 0$ and $y_o = 0$ at $t = 0$, then we have the following equations describing **projectile motion**:

$$x = (v_o\,\cos\alpha_o)\,t \tag{1}$$
$$y = (v_o\,\sin\alpha_o)\,t - \frac{1}{2}gt^2 \tag{2}$$
$$v_x = v_o\,\cos\alpha_o \tag{3}$$
$$v_y = v_o\,\sin\alpha_o - gt \tag{4}$$

The first two equations can be combined to describe the *parabolic* path of the projectile (i.e., the trajectory with no air resistance).

$$y(x) \;=\; (\tan\alpha_o)x \;-\; \frac{g}{2v_o^2\cos^2\alpha_o}x^2 \qquad \text{(the trajectory is a parabola)} \tag{5}$$

A lot of information can be derived from these equations. For example:

The distance from the origin: $r = \sqrt{x^2 + y^2}$

where the position vector is: $\vec{r}(t) \;=\; x(t)\hat{\imath} + y(t)\hat{\jmath}$

The speed: $v = \sqrt{v_x^2 + v_y^2}$

where the velocity vector is: $\vec{v}(t) \;=\; v_x(t)\hat{\imath} + v_y(t)\hat{\jmath}$

where the angle of the velocity vector is: $\tan\alpha = v_y/v_x$

Ex. 16 On level ground a shell is fired with an initial velocity of 40.0 m/s at 60.0^o above the horizontal and feels no appreciable air resistance. (a) Find the horizontal and vertical components of the shell's initial velocity. (b) How long does it take the shell to reach its highest point? (c) Find its maximum height above the ground. (d) How far from its firing point does the shell land? (e) At its highest point, find the horizontal and vertical components of its acceleration and velocity.

Ex. 21 A man stands on the roof of a 15.0-m-tall building and throws a rock with a velocity of magnitude 30.0 m/s at an angle of 33.0^o above the horizontal. You can ignore air resistance. Calculate (a) the maximum height above the roof reached by the rock; (b) the magnitude of the velocity of the rock just before it strikes the ground; and (c) the horizontal range from the base of the building to the point where the rock strikes the ground. (d) Draw $x-t$, $y-t$, v_x-t and v_y-t graphs for the motion.

4 Motion in a Circle

When a particle moves in a circular path, its velocity vector is constantly changing. We will look at the case where the magnitude of the velocity vector is *constant* (i.e., uniform circular motion), and only the direction is changing.

4.1 Uniform Circular Motion

Figure 3.28 Finding the change in velocity, $\Delta\vec{v}$

$$\frac{|\Delta\vec{v}|}{v_1} = \frac{\Delta s}{R} \qquad \text{or} \qquad |\Delta\vec{v}| = \frac{v_1}{R}\Delta s$$

$$a_{av} = \frac{|\Delta\vec{v}|}{\Delta t} = \frac{v_1}{R}\frac{\Delta s}{\Delta t}$$

Calculating the instantaneous velocity we find:

$$a = \lim_{\Delta t\to 0}\frac{v_1}{R}\frac{\Delta s}{\Delta t} = \frac{v_1}{R}\lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t} = \frac{v_1^2}{R}$$

In general, we can write the instantaneous acceleration for uniform circular motion as:

$$a_{\text{rad}} = \frac{v^2}{R} = \frac{4\pi^2 R}{T^2} \qquad \text{(uniform circular motion)} \tag{6}$$

where $v = 2\pi R/T$ and T is the *period*.

Figure 3.29(a) Finding the change in velocity, $\Delta\vec{v}$

Ex. 23 The earth has a radius of 6380 km and turns around once on its axis in 24 h. a) What is the radial acceleration of an object at the earth's equator? Give your answer in m/s^2 and as a fraction of g. b) If a_{rad} at the equator is greater than g, objects would fly off the earth's surface and into space. (We will see the reason for this in Chapter 5.) What would the period of the earth's rotation have to be for this to occur?

4.2 Non-uniform Circular Motion

Non-uniform circular motion occurs when there is tangential acceleration. To differentiate this tangential acceleration from the centripetal acceleration (a_{rad}), we define the following two accelerations:

$$a_{\text{rad}} = \frac{v^2}{R} \qquad \text{and} \qquad a_{\tan} = \frac{d\,|\vec{v}|}{dt}$$

Figure 3.30 Non-uniform circular motion with both *tangential* and *radial* acceleration

5 Relative Velocity

In this section we will determine how the velocity vector changes depending on the reference frame (sometimes called the *inertial* frame) from which it is measured. In other words, if the velocity vector is measured in one reference frame, how does it appear in another reference frame. In all cases, the reference frames are assumed to move with *constant velocity*.

5.1 Relative Velocity in One Dimension

Let's take a look at a reference frame in *one dimension* where an event $(x,\, t)$ is measured in reference frame A. Let's call this measurement $(x_A,\, t_A)$. How does this event appear in reference frame B where $v_{B/A}$ is the velocity of reference frame B with respect to reference frame A?

Figure 3.32 The position of an event as observed from reference frame A and reference frame B

$$x_{P/A} = x_{P/B} + x_{B/A} \qquad \text{and} \qquad \frac{dx_{P/A}}{dt} = \frac{dx_{P/B}}{dt} + \frac{dx_{B/A}}{dt} \tag{7}$$

or

$$v_{P/A} = v_{P/B} + v_{B/A}$$

where, again, $v_{B/A}$ is the velocity of reference frame B with respect to reference frame A.

Ex. 31 A "moving sidewalk" in an airport terminal building moves at 1.0 m/s and is 35.0 m long. If a woman steps on at one end and walks at 1.5 m/s relative to the moving sidewalk, how much time does she require to reach the opposite end if she walks a) in the same direction the sidewalk is moving? b) in the opposite direction?

5.2 Relative Velocity in Two or Three Dimensions

Now we can expand our one-dimensional equation (Eq. 7) into 2 and 3 dimensions by writing the same equation in vector notation.

$$\vec{r}_{P/A} = \vec{r}_{P/B} + \vec{r}_{B/A} \qquad \text{and} \qquad \frac{d\vec{r}_{P/A}}{dt} = \frac{d\vec{r}_{P/B}}{dt} + \frac{d\vec{r}_{B/A}}{dt} \tag{8}$$

or

$$\vec{v}_{P/A} = \vec{v}_{P/B} + \vec{v}_{B/A}$$

Prob. 34 The nose of an ultralight plane is pointed due south, and its air-speed indicator shows 35 m/s. The plane is in a 10-m/s wind blowing toward the southwest relative to the earth. (a) In a vector-addition diagram, show the relationship of $\vec{v}_{P/E}$ (the velocity of the plane relative to the earth) to the two given vectors. (b) Let x be east and y be north, and find the components of $\vec{v}_{P/E}$.

Figure 3.34 A woman walking across a railroad car while traveling down the track

Figure 3.36 The ground speed of an airplane as determined from the air-speed and wind velocity

Prob. 64 A 2.7-kg ball is thrown upward with an initial speed of 20.0 m/s from the edge of a 45.0-m-high cliff. At the instant the ball is thrown, a woman starts running away from the base of the cliff with a constant speed of 6.00 m/s. The woman runs in a straight line on level ground. Ignore air resistance on the ball. (a) At what angle above the horizontal should the ball be thrown so that the runner will catch it just before it hits the ground, and how far does she run before she catches the ball? (b) Carefully sketch the ball's trajectory as viewed by (i) a person at rest on the ground, and (ii) the runner.