

Chapter 1

The Bohr Atom

1 Introduction

Niels Bohr was a Danish physicist who made a fundamental contribution to our understanding of atomic structure and quantum mechanics. He made the first successful attempt at modeling the hydrogen atom by assuming that the electron executes orbital motion about the proton (i.e., the nucleus). Assuming that the only force between the electron and the proton is the electrostatic force, and applying Newton's 2nd law, he arrived at the following equation:

$$\sum F = ma \quad \rightarrow \quad \frac{e^2}{4\pi\epsilon_0 r^2} = m \frac{v^2}{r} \quad (1)$$

where m , v , and r , are the mass, velocity and radius of the electron moving in a circular orbit about the proton. Note: we assume that $m_e/m_p \ll 1$, such that position of the proton remain essentially motionless. Simplifying Eq. 1, we find the following:

$$\frac{e^2}{4\pi\epsilon_0} = (mvr) v \quad (2)$$

Up until this point, we have applied only classical physics. Furthermore, classical physics would predict that this simple planetary model would cause the electron to continually emit its kinetic energy until the electron's orbit completely collapses into the proton. A new assumption must be added to this model in order to keep the atom stable, otherwise, we would not be here. At this point, Bohr made a significant contribution by concluding that the electrons would not have to radiate their kinetic energy if the orbit of the electron were such that its angular momentum L is quantized.

$$L = mvr = n\hbar \quad (\text{Bohr's postulate}) \quad (3)$$

where $n = 1, 2, 3, \dots$, and \hbar is equal to Planck's constant h divided by 2π . Planck's constant is $h = 6.626 \times 10^{-34}$ J·s.

2 Key features of the Bohr Model

Using the model described in the introduction, we are prepared to calculate some key features of the hydrogen atom and compare them with experimental measurements. Substituting Eq. 3 into Eq. 2, we obtain the following:

$$\frac{e^2}{4\pi\epsilon_o} = (n\hbar) v_n \quad \rightarrow \quad v_n = \left(\frac{e^2}{4\pi\epsilon_o\hbar} \right) \frac{1}{n}$$

By multiplying the numerator and denominator by c , the speed of light, we obtain the following equation for v_n :

$$v_n = \left(\frac{e^2}{4\pi\epsilon_o\hbar c} \right) \frac{c}{n} = \alpha \frac{c}{n} \quad (4)$$

where α is a *dimensionless* constant called the *fine structure* constant, $\sim 1/137$. Thus, an electron in the $n = 1$ state (also called the *ground state*), has a velocity of $c/137$, or approximately $\sim 1\%$ of the speed of light. Note, as the principle quantum number n increases, the speed of the electron decreases.

Substitute v_n from Eq. 4 into Eq. 2, we find that

$$\begin{aligned} \frac{e^2}{4\pi\epsilon_o} &= (mv_n r_n) v_n \quad \rightarrow \quad \left(\frac{e^2}{4\pi\epsilon_o\hbar c} \right) \hbar c = \left(m \frac{\alpha c}{n} r_n \right) \frac{\alpha c}{n} \\ \alpha \hbar c &= \frac{m c^2 \alpha^2 r_n}{n^2} \quad \rightarrow \quad r_n = \frac{n^2 \hbar}{m \alpha c} \end{aligned} \quad (5)$$

In the ground state ($n = 1$), the ground-state radius would be

$$r_1 = \frac{1^2 \hbar}{m \alpha c} = 0.0529 \text{ nm} \quad (6)$$

a radius that agrees very well with the radius of the hydrogen atom.

Exercise: Using the following conversion ($1 \text{ eV} = 1.602 \times 10^{-19} \text{ joules}$) and $h = 6.626 \times 10^{-34} \text{ J}\cdot\text{s}$, show that:

- a) $m_e c^2 \simeq 511,000 \text{ eV}$, and
- b) $\hbar c \simeq 197 \text{ eV}\cdot\text{nm}$
- c) $hc \simeq 1240 \text{ eV}\cdot\text{nm}$

You will find that these three definitions are used *over and over* again in atomic physics, so, these are worth remembering. For example, these definitions can be used to simplify the calculation performed in Eq. 6.

Finally, a third parameter that can be calculated using the Bohr model is the *total energy* of the electron as it orbits the proton. This can be written as the sum of the kinetic and potential energies.

$$E = \frac{1}{2}m_e v^2 - \frac{e^2}{4\pi\epsilon_0 r} \quad (7)$$

Using the results for v_n and r_n , we can rewrite Eq. 7 using quantized values:

$$E_n = \frac{1}{2}m_e v_n^2 - \frac{e^2}{4\pi\epsilon_0 r_n} \quad (8)$$

Substituting the results for v_n and r_n , we find that the energies levels are quantized and described by the following equation:

$$E_n = -\frac{1}{2}m_e c^2 \frac{\alpha^2}{n^2} \quad (9)$$

Substituting the value $n = 1$, we can calculate the ground-state energy to be the following:

$$E_1 = -13.6 \text{ eV}$$

Once again, if we compare this results to the ground-state energy of the hydrogen atom, we find it is in excellent agreement.

Question: How well does Eq. 9 describe the light emitted from the hydrogen atom?

To answer this question, let's apply the conservation of energy to this isolated system. Furthermore, let's assume that the electron is initially in the $n = 3$ state and makes a transition to the $n = 2$ state. As part of the transition, a photon of energy hc/λ is emitted in order to conserve the total energy of the system. The conservation of energy can be written as:

$$E_{\text{before}} = E_{\text{after}} \quad \rightarrow \quad E_3 = E_2 + \frac{hc}{\lambda}$$

where λ is the wavelength of the emitted photon.

$$E_3 - E_2 = \frac{hc}{\lambda} \quad \rightarrow \quad -\frac{1}{2}m_e c^2 \frac{\alpha^2}{3^2} - \left(-\frac{1}{2}m_e c^2 \frac{\alpha^2}{2^2}\right) = \frac{hc}{\lambda}$$

$$\frac{1}{2}m_e c^2 \alpha^2 \left(\frac{1}{2^2} - \frac{1}{3^2}\right) = \frac{hc}{\lambda} \quad \rightarrow \quad \frac{5}{72}m_e c^2 \alpha^2 = \frac{hc}{\lambda}$$

or

$$\lambda = \frac{72}{5} \frac{hc}{m_e c^2 \alpha^2}$$

Using the identity $hc = 1240 \text{ eV}\cdot\text{nm}$, we find that the wavelength λ is:

$$\lambda = \frac{72}{5} \frac{1240 \text{ eV}\cdot\text{nm}}{511,000 \text{ eV} \left(\frac{1}{137}\right)^2} = 655.8 \text{ nm}$$

which is very close to the value of 656 nm (red) spectral line measured in the laboratory. This is often referred to as the H_α (H-alpha) line. Here again, we have another confirmed prediction of the Bohr theory!

3 Conclusion

As a result of the Bohr hypothesis ($L = n\hbar$), we have three simple results (i.e., predictions) about the electron as it orbits the proton in the hydrogen atom:

$$v_n = \alpha \frac{c}{n} \quad (\text{velocity}) \quad (10)$$

$$r_n = \frac{n^2 \hbar}{m_e \alpha c} \quad (\text{radius}) \quad (11)$$

$$E_n = -\frac{1}{2} m_e c^2 \frac{\alpha^2}{n^2} \quad (\text{total energy}) \quad (12)$$

While the first prediction is impossible to verify, the other two predictions agree very well with the laboratory measurements made on the hydrogen atom. Furthermore, the last prediction (Eq. 12), along with conservation of energy, describes very well the wavelengths observed from the hydrogen atom as electrons cascade downward from higher n states to lower n states.