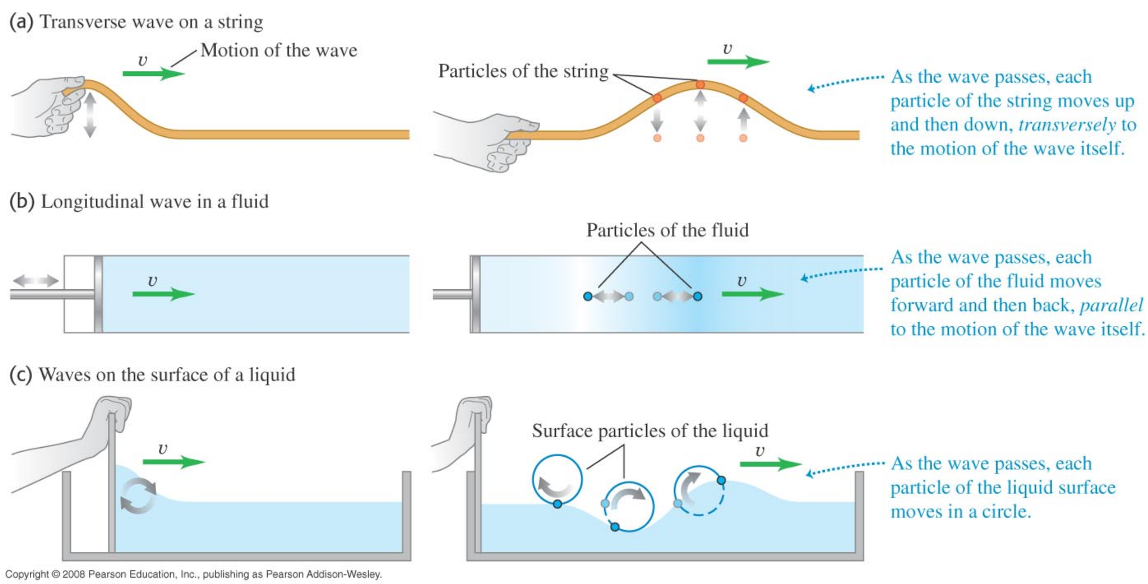
Chapter 15
Mechanical Waves

1 Types of Mechanical Waves

This chapter and the next are about mechanical waves–waves that travel within some material called a medium. Waves play an important role in how we perceive our physical world (e.g., sight and sound). Not all waves are mechanical, (e.g., electromagnetic waves); however, the terminology and concepts we introduce in this chapter apply to all kinds of waves.

A mechanical wave is a disturbance that travels through some material or substance called the medium for the wave. There are basically two kinds of waves–**transverse** and **longitudinal** waves.



Waves propagate through the medium at a definite speed called the **wave speed**. As the wave propagates through the medium, the particles sustaining the wave motion move in *simple harmonic motion* about their equilibrium points. While there is no net transport of matter during the propagation of a wave, there is *energy transport* from one region to another.

2 Periodic Waves

As the name implies, these waves occur with a certain repetition that can be described characteristic properties.

A = the amplitude (mm, or m)
f = the frequency (Hz)
T = the period (s)
λ = the wavelength (mm, or m)

A special class of periodic waves are harmonic waves. These waves can be described by *sine* and *cosine* functions, similar to what we did with simple harmonic motion. For periodic waves, there is a relation between the wavelength and frequency:

$$v = f\lambda$$

where v is the speed of propagation.

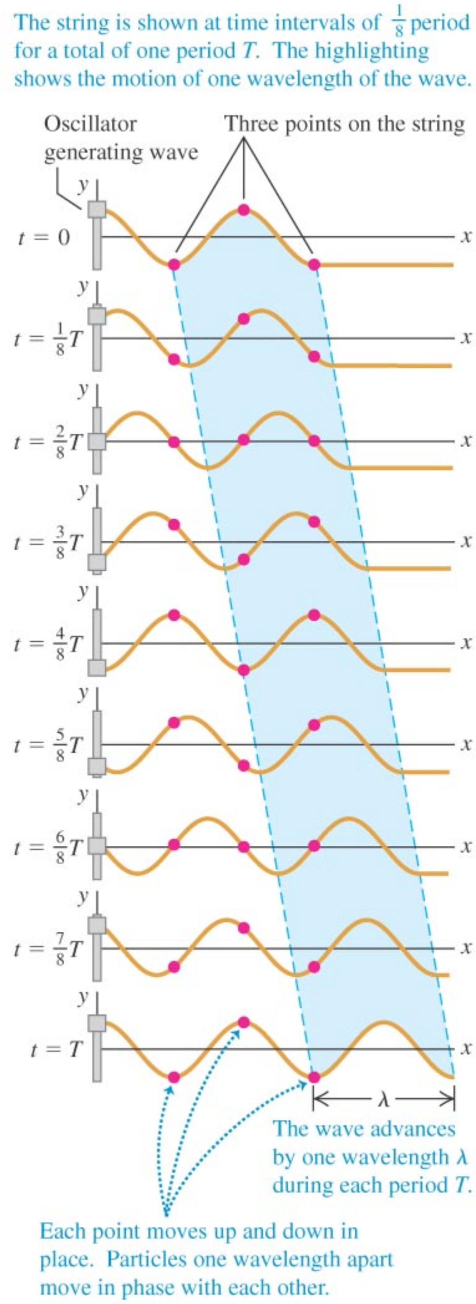
2.1 Transverse Waves

Particles in the medium move in a direction perpendicular to the direction of propagation. Waves generated by "shaking" a taut string "up and down" will generate transverse waves. Interestingly, some transverse waves do not require a medium to transport energy. As an example, electromagnetic waves can travel in a vacuum and they also carry energy.

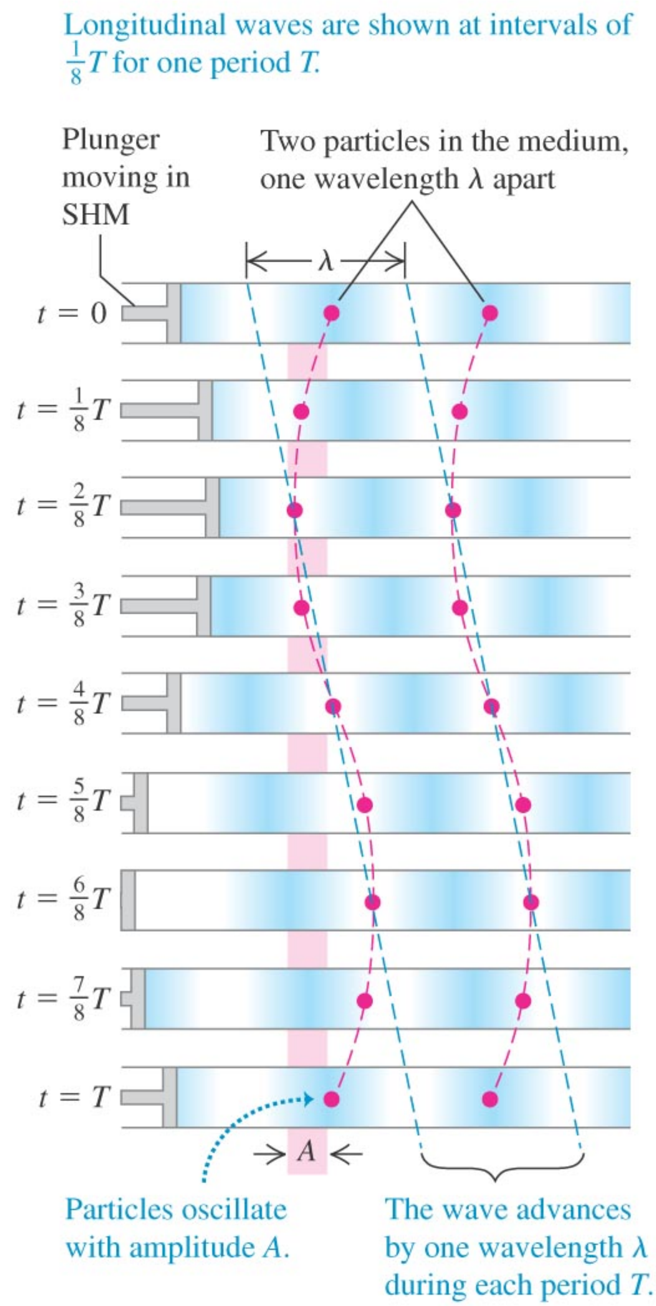
2.2 Longitudinal Waves

Particles in the medium move in a direction parallel and anti-parallel to the direction of propagation. Sound is an example of longitudinal waves.

Ex. 1 The speed of sound in air at 20°C is 344 m/s. (a) What is the wavelength of a sound wave with a frequency of 784 Hz, corresponding to the note G_5 on a piano, and how many milliseconds does each vibration take? (b) What is the wavelength of a sound wave one octave higher than the note in part (a).



The string is shown at time intervals of $\frac{1}{8}$ period for a total of one period T. The highlighting shows the motion of one wavelength of the wave.
Oscillator generating wave
Three points on the string
$t = 0$
$t = \frac{1}{8}T$
$t = \frac{2}{8}T$
$t = \frac{3}{8}T$
$t = \frac{4}{8}T$
$t = \frac{5}{8}T$
$t = \frac{6}{8}T$
$t = \frac{7}{8}T$
$t = T$
y
x
λ
The wave advances by one wavelength λ during each period T.
Each point moves up and down in place. Particles one wavelength apart move in phase with each other.


Longitudinal waves are shown at intervals of $\frac{1}{8}T$ for one period T.
Plunger moving in SHM
Two particles in the medium, one wavelength λ apart
λ
$t = 0$
$t = \frac{1}{8}T$
$t = \frac{2}{8}T$
$t = \frac{3}{8}T$
$t = \frac{4}{8}T$
$t = \frac{5}{8}T$
$t = \frac{6}{8}T$
$t = \frac{7}{8}T$
$t = T$
A
Particles oscillate with amplitude A.
The wave advances by one wavelength λ during each period T.

3 Mathematical Description of a Wave

We introduce the concept of a **wave function** $y = y(x,t)$ in order to quantify the properties of wave motion. Once we know the wave function, we can calculate the velocity, acceleration, and the amplitude of the wave at all times.

3.1 Wave Function for a Harmonic Wave

A sinusoidal wave is the simplest way to describe the motion of a string (or medium) as it oscillates about its equilibrium position when a harmonic wave is being propagated. Let's write the wave function for a wave traveling from *left-to-right* and see how it works:

$$y(x,t) \;=\; A\cos 2\pi f\left(\frac{x}{v} - t\right) \;=\; A\cos 2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)$$

However, there is a more convenient form for writing a sinusoidal wave traveling *left-to-right*:

$$y(x,t) \;=\; A\cos\left(kx - \omega t\right) \qquad \textbf{(left-to-right)}$$

where $k = 2\pi/\lambda$ is called the **wave number**, and $\omega = 2\pi f$ is the angular frequency.

To write a traveling wave that is moving in the opposite direction (*right-to-left*), all we have to do is change the $-$ sign into a $+$ sign in the following way:

$$y(x,t) \;=\; A\sin\left(kx + \omega t\right) \qquad \textbf{(right-to-left)}$$

Ex. 8 A certain transverse wave is described by

$$y(x,t) \;=\; (6.50\,\mathrm{mm})\ \cos\,2\pi\left(\frac{x}{28.0\,\mathrm{cm}} - \frac{t}{0.0360\,\mathrm{s}}\right)$$

Determine the wave's a) amplitude; b) wavelength; c) frequency; d) speed of propagation; e) direction of propagation.

3.2 Particle Velocity and Acceleration in a Sinusoidal Wave

Again, once you know the wave function $y(x,t)$, you can calculate the velocity and acceleration of the particle. Let's examine the transverse velocity and acceleration of a traveling, harmonic wave moving from left-to-right:

$$\begin{aligned} y(x,t) &= A\cos(kx-\omega t) \\ v_y(x,t) &= \frac{\partial y(x,t)}{\partial t} = \omega A\sin(kx-\omega t) \\ a_y(x,t) &= \frac{\partial^2 y(x,t)}{\partial t^2} = -\omega^2 A\cos(kx-\omega t) \end{aligned}$$

3.3 The Wave Equation

Calculating the second-order derivatives of the wave function $y(x,t)$ and using the relation $v=\omega/k$, we find the following:

$$\frac{\partial^2 y(x,t)}{\partial x^2} = -k^2\, y(x,t) \tag{1}$$

$$\frac{\partial^2 y(x,t)}{\partial t^2} = -\omega^2\, y(x,t) \tag{2}$$

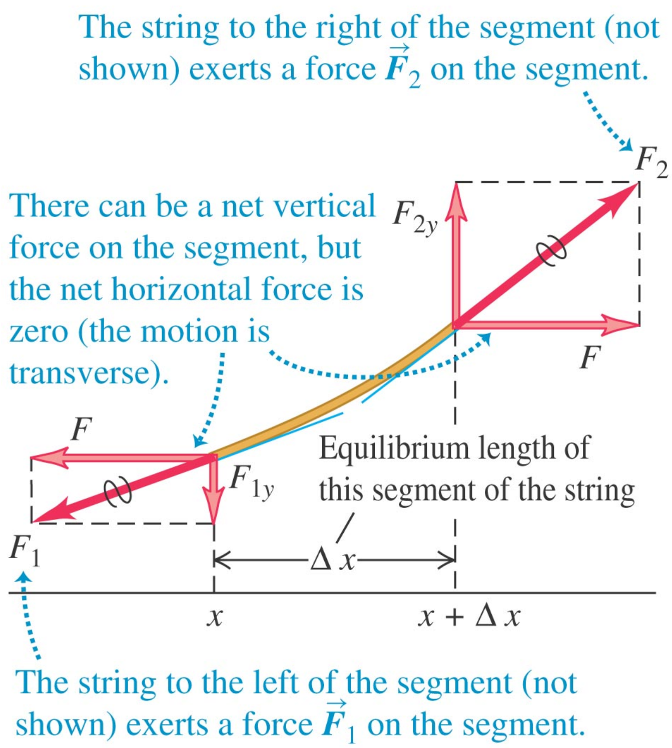
Combining the above two equations, we find:

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2} \qquad \text{(The Wave Equation)} \tag{3}$$

While this equation was derived using traveling, harmonic waves, it is true for other waves as well. For example, see if the following wave function $y(x,t)=A/(x-vt)^2$ satisfies the wave equation.
What about this wave function: $y(x,t)=A/(x^2-v^2t^2)$?

4 Speed of a transverse Wave

We apply Newton's second law, $\sum\vec{F}=m\vec{a}$, to a small segment of strength whole leng in the equilibrium position is Δx.



$$\frac{F_{1y}}{F} = -\left(\frac{\partial y}{\partial x}\right)_x \qquad\qquad \frac{F_{2y}}{F} = \left(\frac{\partial y}{\partial x}\right)_{x+\Delta x}$$

The net force in the y direction is:

$$F_y = F_{1y} + F_{2y} = F\left[\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right]$$

Let's insert this equation into Newton's second law, $\sum F_y = ma_y$:

$$F\left[\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right] = (\mu\,\Delta x)\,\frac{\partial^2 y}{\partial t^2} \tag{4}$$

The left-hand side of this equation looks very similar to the definition for the second-order derivative, $\partial^2 y/\partial x^2$. In fact, using the *fundamental definition of a derivative*, we can write:

$$\frac{\partial^2 y}{\partial x^2} \equiv \lim_{\Delta x \to 0} \frac{\left[\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right]}{\Delta x} \tag{5}$$

Now, we can rewrite Eq. 4 using the definition from Eq. 5:

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2}$$

Using the *wave equation* (Eq. 3), we can find the propagation velocity v:

$$v = \sqrt{\frac{F}{\mu}} \qquad \text{(velocity of propagation)}$$

Ex. 15 One end of a horizontal rope is attached to a prong of an electrically driven tuning fork that vibrates at 120 Hz. The other end passes over a pulley and supports a 1.50-kg mass. The linear mass density of the rope is 0.0480 kg/m. a) What is the speed of a transverse wave on the rope? b) What is the wavelength? c) How would your answers to parts (a) and (b) change if the mass were increase to 3.00 kg?

5 Energy in Wave Motion

Every wave motion has *energy* associated with it. Let's look at a transverse string and investigate how the energy is transferred from one portion of a string to another. Let's assume we have a sinusoidal wave traveling from left-to-right.

The y-component of force is proportional to the negative of the slope of the wave function. Draw a picture of this to convince yourself it's true.

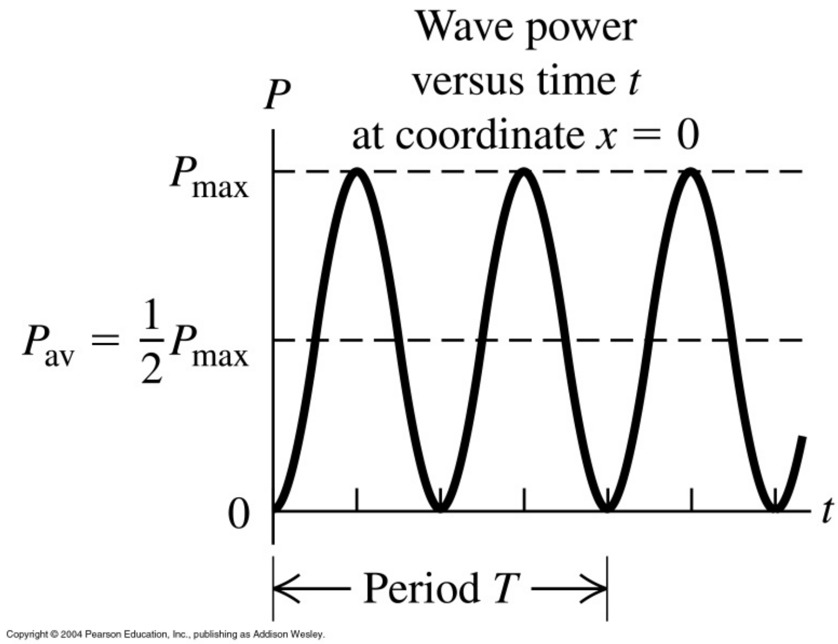
$$F_y(x,t) = -F\frac{\partial y(x,t)}{\partial x}$$

As we've seen in a previous chapter, the power is the rate at which work is done, or power $= P = F\,\Delta s/\Delta t$.

$$P_{\text{inst}} = \frac{dW}{dt} = F_y(x,t)\,v_y(x,t) = -F\,\frac{\partial y(x,t)}{\partial x}\,\frac{\partial y(x,t)}{\partial t}$$

$$P(x,t) = Fk\omega A^2\sin^2(kx-\omega t) \qquad P(x,t) = \sqrt{\mu F}\,\omega^2 A^2\sin^2(kx-\omega t)$$

$$P_{\max} = \sqrt{\mu F}\,\omega^2 A^2 \qquad\qquad P_{\text{av}} = \frac{1}{2}\sqrt{\mu F}\omega^2 A^2$$



Ex. 22 A piano wire with mass 3.00 g and length 80.0 cm is stretched with a tension of 25.0 N. A wave with frequency 120.0 Hz and amplitude 1.6 mm travels along the wire. a) Calculate the average power carried by the wave. b) What happens to the average power if the wave amplitude is halved?

5.1 Wave intensity

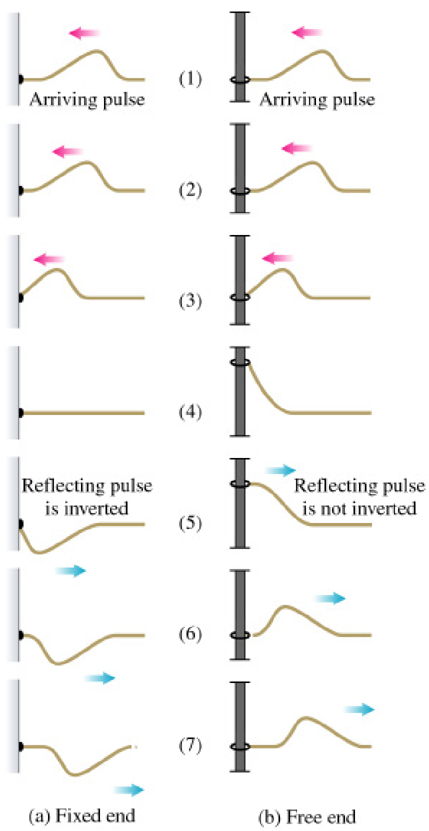
If the energy is propagated from the source in three dimensions, then the intensity becomes the quantity of interest. The *intensity* is defined to be the *Power/Area*

$$I = \frac{\text{Power}}{\text{Area}} = \frac{P}{4\pi R^2} \qquad \text{(the intensity)}$$

and has units of *watts/m*^{2}.

6 Wave Interference, Boundary Conditions, and Superposition

What happens when a wave meets a boundary? When a wave meets a boundary there are two extreme cases to consider. The first is the *fixed end* boundary where the wave is reflected as in *inverted* wave. On the other hand, if the wave encounters a *free end* boundary, the wave is reflected and *not* inverted.



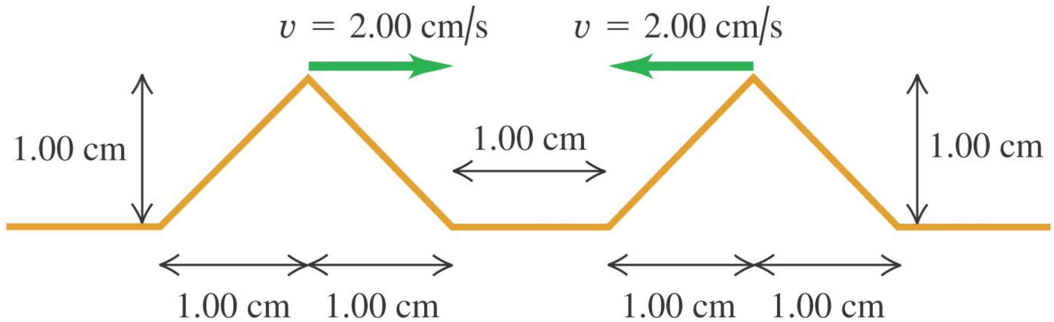
6.1 The Principle of Superposition

The principle of superposition states that we can combine the displacements of separate pulses (i.e., waves) at each point to obtain the actual displacement.

"When two waves overlap, the actual displacement of any point on the string at any time is obtained by adding the displacement the point would have if only the first wave were present and the displacement it would have if only the second wave were present."

$$y(x,t) \;=\; y_1(x,t) + y_2(x,t) \qquad \text{(The Principle of Superposition)}$$

Ex. 30 Interference of Triangular Pulses. Two triangular wave pulses are traveling toward each other on a stretched string as shown in Fig. E15.32. Each pulse is identical to the other and travels at 2.00 cm/s. The leading edges of the pulses are 1.00 cm apart at $t = 0$. Sketch the shape of the string at $t = 0.250$ s, $t = 0.500$ s, $t = 0.750$ s, $t = 1.000$ s, and $t = 1.250$ s.



7 Standing Waves on a String

As an example of superposition, let's look at standing waves on a string. This can be viewed as the superposition of two traveling waves with the same amplitude, frequency, and wavelength–one wave traveling *left* → *right*, and the second wave traveling *right* → *left*.

$$y_1 \;=\; A\cos(kx - \omega t) \qquad (\mathrm{L} \rightarrow \mathrm{R})$$
$$y_2 \;=\; -A\cos(kx + \omega t) \qquad (\mathrm{R} \rightarrow \mathrm{L})$$

The combined wave functions, using the principle of superposition, is:

$$y(x,t) \;=\; y_1(x,t) + y_2(x,t) = (2A\,\sin kx)\,\sin\omega t$$

If we define the amplitude of the standing wave $A_{\mathrm{SW}} = 2A$, we can rewrite the above equation as:

$$y(x,t) = (A_{\mathrm{SW}} \sin kx) \sin \omega t \qquad \text{(Equation for a Standing Wave)}$$

Notice that at $t = 0$, we have complete cancellation of the combined wave functions.

Where do the nodes of the standing wave occur? They occur where $\sin kx = 0$.

$$x = 0, \ \frac{\lambda}{2}, \ \frac{2\lambda}{2}, \ \frac{3\lambda}{2}, \ \cdots$$

Ex. 37 Standing waves on a wire are described by Eq. (15.28), with A_{SW}=2.50 mm, $\omega = 942$ rad/s, and $k = 0.750\pi$ rad/m. The left end of the wire is at $x = 0$. At what distances from the left end are (a) the nodes of the standing wave and (b) the antinodes of the standing wave?

8 Normal Modes of a String

In the previous section, we investigated the *standing waves* on an arbitrarily long string. Because the string could be any length, we could generate identical opposing traveling waves of any desirable wavelength, thus producing *standing waves* of any wavelength.

In this section, we will investigate the standing waves produced on a fixed length string, let's say length = L. As we will see, this will result in wavelengths of particular lengths labeled by an index n.

Let's calculate the wavelengths of standing waves that can be supported on a string of length L when both ends are fixed.

$$L = n\frac{\lambda}{2} \qquad (\text{n} = 1,\ 2,\ 3,\ .\ .\ .\ .\ )$$

$$\lambda_n = \frac{2L}{n} \qquad (\text{n} = 1,\ 2,\ 3,\ .\ .\ .\ .\ )$$

$n = 1$	fundamental	1^{st} harmonic
$n = 2$	1^{st} overtone	2^{nd} harmonic
$n = 3$	2^{nd} overtone	3^{rd} harmonic
$n = 4$	3^{rd} overtone	4^{th} harmonic

Using the relationship $v = f_{\text{n}}\lambda_{\text{n}}$, we can write the frequency of the n^{th} wavelength as:

$$f_{\text{n}} = \frac{nv}{2L} = nf_1 \qquad (\text{n} = 1,\, 2,\, 3,\, \dots) \tag{6}$$

where:

For a string of length L, we have the following wave function describing the possible standing waves:

$$y_{\text{n}}(x,t) = A_{\text{SW}} \sin k_{\text{n}} x \sin \omega_{\text{n}} t$$

where $k_{\text{n}} = 2\pi/\lambda_{\text{n}}$, and $\omega_{\text{n}} = 2\pi f_{\text{n}}$.

If the tension F and mass density μ is known, then Eq. 6 can be rewritten as:

$$f_{\text{n}} = \frac{n}{2L}\sqrt{\frac{F}{\mu}} \qquad \text{(standing waves on a fixed length string)}$$

Ex. 40 A piano tuner stretches a steel piano wire with a tension of 800 N. The steel wire is 0.400 m long and has a mass of 3.00 g. a) What is the frequency of its fundamental mode of vibration? b) What is the number of the highest harmonic that could be heard by a person who is capable of hearing frequencies up to 10,000 Hz.

Ex. 47 **Guitar String.** One of the 63.5 cm-long strings of an ordinary guitar is tuned to produce the note B_3 (frequency 245 Hz) when vibrating in its fundamental mode. a) Find the speed of transverse waves on this string. b) If the tension in this string is increased by 1.0%, what will be the new fundamental frequency of the string? c) If the speed of sound in the surrounding air is 344 m/s, find the frequency and wavelength of the sound wave produced in the air by the vibration of the B_3 string. How do these compare to the frequency and wavelength of the standing wave on the string?

Prob. 48 "Not part of the homework." A transverse wave on a rope is given by

$$y(x,t) = (0.75\ \text{cm}) \cos \pi \left[\left(0.400\ \text{cm}^{-1}\right) x + \left(250\ \text{s}^{-1}\right) t\right]$$

a) Find the amplitude, period, frequency, wavelength, and speed of propagation. b) Sketch the shape of the rope at these values of t: 0, 0.0005 s, 0.0010 s. c) Is the wave traveling in the $+x$- or $-x$-direction? d) The mass per unit length of the rope is 0.0500 kg/m. Find the tension. e) Find the average power of this wave.

Prob. 71 A large rock that weighs 164.0 N is suspended from the lower end of a thin wire that is 3.00 m long. The density of the rock is 3200 kg/m^3. The mass of the wire is small enough that its effect on the tension in the wire can be ignored. The upper end of the wire is held fixed. When the rock is in air, the fundamental frequency for transverse standing waves on the wire is 42.0 Hz. When the rock is totally submerged in a liquid, with the top of the rock just below the surface, the fundamental frequency for the wire is 28.0 Hz. What is the density of the liquid?