Chapter 10
Dynamics of Rotational Motion

1 Torque

In this chapter we will investigate how the combination of *force* (F) and the *moment arm* (ℓ) effect a change in rotational motion (i.e., rotational angular acceleration, α).

$$\tau = \text{force} \times \text{moment arm} \qquad \text{(definition of torque)}$$

where the moment arm is the distance of *closest approach* to the *line of action* of the force.

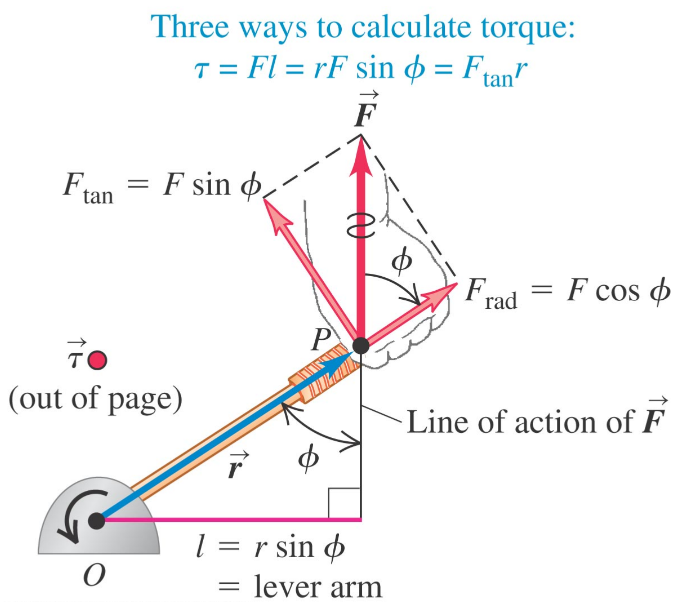


Figure 1: Figure 10.3 from University Physics 15th edition.

A *positive* torque is one that causes an object in the *x*-*y* plane to rotate in the *counter-clockwise* direction, while *negative* torques are those the cause an object to rotate in the *clockwise* direction.

clockwise motion (-θ) $\rightarrow$ negative torque

counter-clockwise motion (+θ) $\rightarrow$ positive torque

The magnitude of the torque can also be written as:

$$\tau = F\,\ell = rF\,\sin\phi$$

The vector definition of the torque can be written as:

$$\vec{\tau} = \vec{r} \times \vec{F} \qquad \text{(the torque vector)}$$

How is the angle ϕ defined? It is the angle between $\vec{r}$ and $\vec{F}$.

Recall how the vector cross-product is defined for $\vec{r} \times \vec{F}$ in chapter 1, equation 1.26.

Ex. 1 Calculate the torque (magnitude and direction) about point O due to the force $\vec{F}$ in each of the situations sketched in **Fig. E10.1**. In each case, the force $\vec{F}$ and the rod both lie in the plane of the page, the rod has length 4.00 m, and the force has magnitude $F = 10.0$ N.

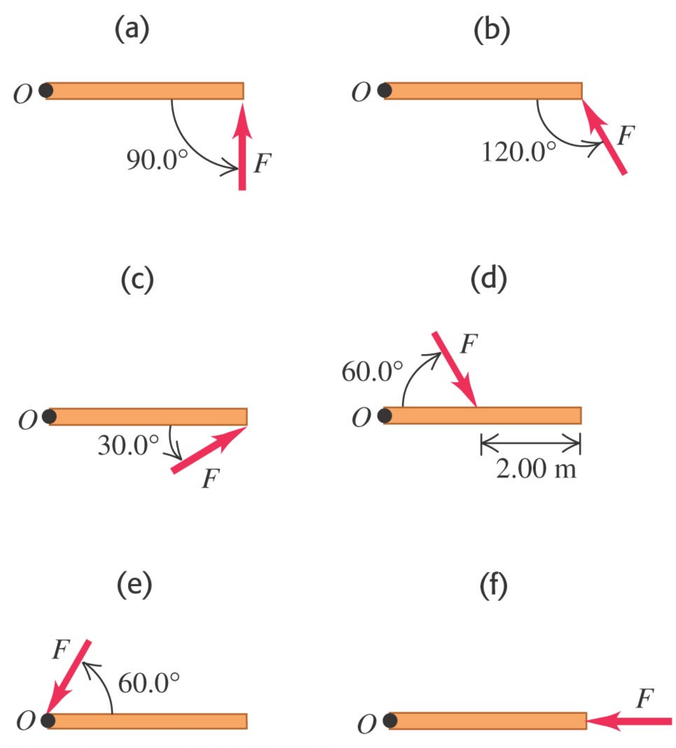


Figure 2: Figure E10.1 from University Physics 15^{th} edition.

2 Torque and Angular Acceleration for a Rigid Body

How is the torque related to the angular acceleration of a rigid body? We will show that the angular acceleration of a rotating rigid body is directly proportional to the sum of the torque components along (i.e., tangential) to the axis of rotation.

$$F_{1,\tan} = m_1\, a_{1,\tan} \qquad \rightarrow \qquad F_{1,\tan} = m_1\, a_{1,\tan} = m_1\,(r_1\,\alpha_z)$$

$$\tau_{1z} = F_{1,\tan}\, r_1 = m_1\, r_1^2\, \alpha_z = I_1\, \alpha_z$$

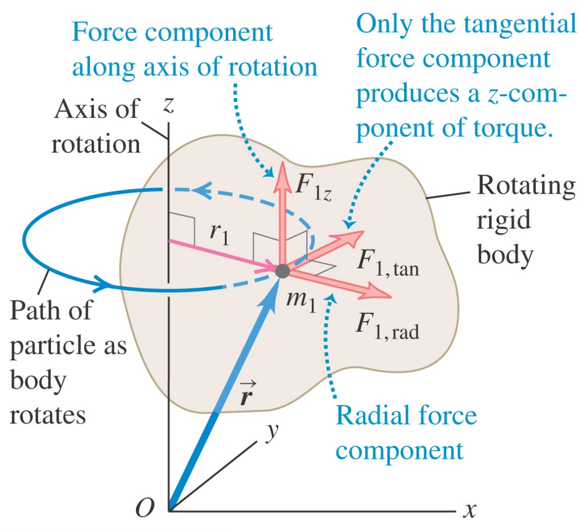


Figure 3: Figure 10.6 from University Physics 15th edition.

$$\tau_z = \tau_{1z} + \tau_{2z} + \tau_{3z} + \cdots = m_1 r_1^2 \alpha_z + m_2 r_2^2 \alpha_z + m_3 r_3^2 \alpha_z + \cdots = I_1\alpha_z + I_2\alpha_z + I_3\alpha_z + \cdots$$

$$\tau_z = \sum \tau_{iz} = \left(\sum m_i r_i^2\right) \alpha_z = I\,\alpha_z \qquad \text{(Newton's 2}^{\text{nd}}\text{ Law)}$$

Ex. 11 A machine part has the shape of a solid uniform sphere of mass 225 g and diameter 3.00 cm. It is spinning about a frictionless axle through its center, but at one point on its equator it is scraping against metal, resulting in a friction force of 0.0200 N at that point. (a) Find its angular acceleration. (b) How long will it take to decrease its rotational speed by 22.5 rad/s?

3 Rigid-Body Rotation About a Moving Axis

3.1 Combined Translation and Rotation: Energy Relations

In this section we extend our definition of kinetic energy K to include both translational and rotational kinetic energy. The *kinetic energy* of a point mass m_i is:

$$K_i = \frac{1}{2}m_i v_i^2 = \frac{1}{2}m_i \vec{v}_i \cdot \vec{v}_i = \frac{1}{2}m_i\left(\vec{v}_{\text{cm}} + \vec{v}_i{}'\right) \cdot \left(\vec{v}_{\text{cm}} + \vec{v}_i{}'\right)$$

The *kinetic energy* of a sum of point masses comprising an extended object as shown in the figure below can be written as:

$$K = \sum_{i=1}^{N} K_i = \sum\left(\frac{1}{2}m_i v_{\text{cm}}^2\right) + \sum\left(m_i \vec{v}_{\text{cm}} \cdot \vec{v}_i{}'\right) + \sum\left(\frac{1}{2}m_i v_i'^2\right)$$

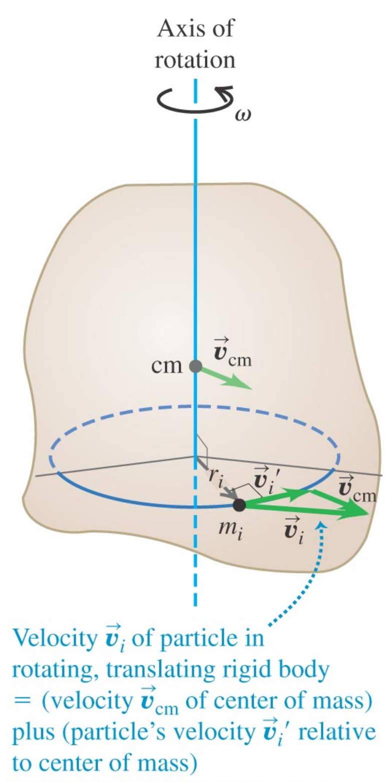


Figure 4: Figure 10.12 from University Physics 15$^{\text{th}}$ edition.

$$K = \frac{1}{2}\left(\sum m_i\right) v_{\text{cm}}^2 + \vec{v}_{\text{cm}} \cdot \underbrace{\left(\sum m_i \vec{v}_i{}'\right)}_{=0} + \sum \left(\frac{1}{2} m_i v_i'^2\right)$$

$$K = K_{\text{trans}} + K_{\text{rot}} = \frac{1}{2} M v_{\text{cm}}^2 + \frac{1}{2} I_{\text{cm}} \omega^2$$

Rolling without slipping:
In the case where we have rolling without slipping

$$v_{\text{cm}} = R\,\omega$$

$$a_{\text{cm}} = R\,\alpha$$

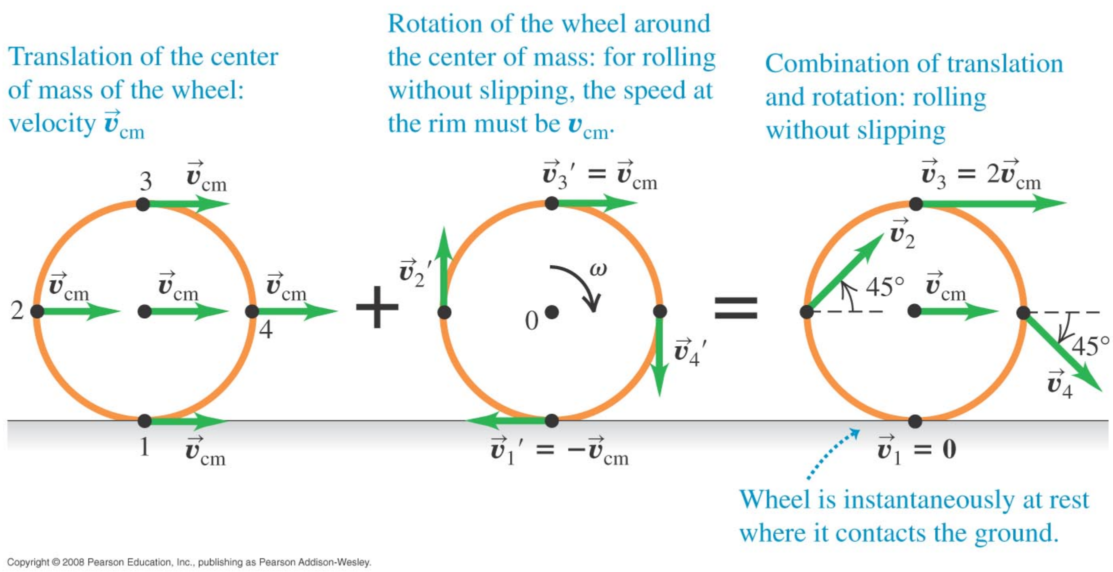


Figure 5: Figure 10.13 from University Physics 15$^{\text{th}}$ edition.

3.2 Combined Translation and Rotation: Dynamics

Both forms of Newton's 2$^{\text{nd}}$ law apply to objects executing both translational and rotational motion.

$$\sum \vec{F}_{\text{ext}} = m\,\vec{a} \tag{1}$$

$$\sum \tau_z = I_{\text{cm}}\,\alpha_z \tag{2}$$

Equation 2 is valid even if the axis of rotation moves, provided the following two conditions are met:

1. the axis through the center of mass must be an axis of symmetry,
2. the axis must not change direction.

Ex. 24 A hollow, spherical shell with mass 2.00 kg rolls without slipping down a 38.0° slope. a) Find the acceleration, the friction force, and the minimum coefficient of friction needed to prevent slipping. b) how would your answers to part (a) change if the mass were doubled to 4.00 kg

3.3 Instantaneous axis of rotation

We can save ourselves a little work if we modify item 1 above to include axes that are parallel to the axis of symmetry. If we invoke the *parallel axis theorem*, we can avoid solving two equations to find the acceleration $a_{\rm cm}$. The acceleration of the *instantaneous* axis of rotation is also the the center-of-mass.

Ex. 30* **A Ball Rolling Uphill.** A bowling ball rolls without slipping up a ramp that slopes upward at an angle β to the horizontal (see Example 10.7 in section 10.3). Treat the ball as a uniform, solid sphere, ignoring the finger holes. a) Draw the free-body diagram for the ball. Explain why the friction force must be directed *uphill*. b) What is the acceleration of the center of mass of the ball? c) What minimum coefficient of static friction is needed to prevent slipping.

4 Work and Power in Rotational Motion

The work performed on a rigid body is due to the force (torque) applied tangentially and displacing it a distance ds.

$$dW = F_{\rm tan}\, ds = F_{\rm tan}\, R\, d\theta = \tau_z\, d\theta$$

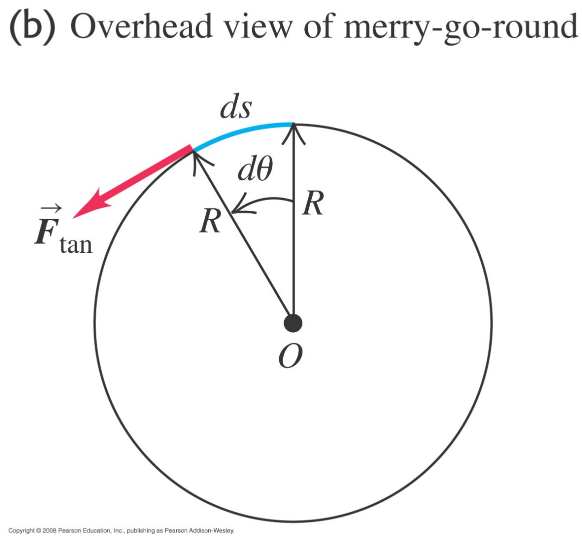


Figure 6: Figure 10.21 (b) from University Physics 15th edition.

The total work done by a torque τ_z during an angular displacement from $\theta_1 \rightarrow \theta_2$ is:

$$W = \int_{\theta_1}^{\theta_2} \tau_z \, d\theta$$

The work done by a constant torque is:

$$W = \tau_z \, \Delta\theta \qquad \text{(work done by a constant torque)}$$

The **work-energy** theorem states that the net work done on an object results in a change in kinetic energy. For rotational motion we can write that the work done by an external torque is:

$$W_{\text{rot}} = \Delta K = K_{\text{f}}^{\text{rot}} - K_{\text{i}}^{\text{rot}} = \frac{1}{2} I \omega_f^2 - \frac{1}{2} I \omega_i^2$$

Rotational Power

The average rotational power due to a constant torque τ_z is:

$$\overline{\text{Power}} = \frac{\Delta W}{\Delta t} = \frac{\tau_z \, \Delta\theta}{\Delta t} = \tau_z \bar{\omega} \qquad \text{(average rotational power)}$$

while the instantaneous power due to a time-varying torque τ_z is:

$$\text{Power} = \frac{dW}{dt} = \frac{\tau_z\, d\theta}{dt} = \tau_z \omega \qquad \text{(instantaneous rotational power)}$$

Ex. 32 An engine delivers 175 hp to the propeller at 2400 rev/min. a) How much torque does the aircraft engine provide? b) How much work does the engine do in one revolution of the propeller.

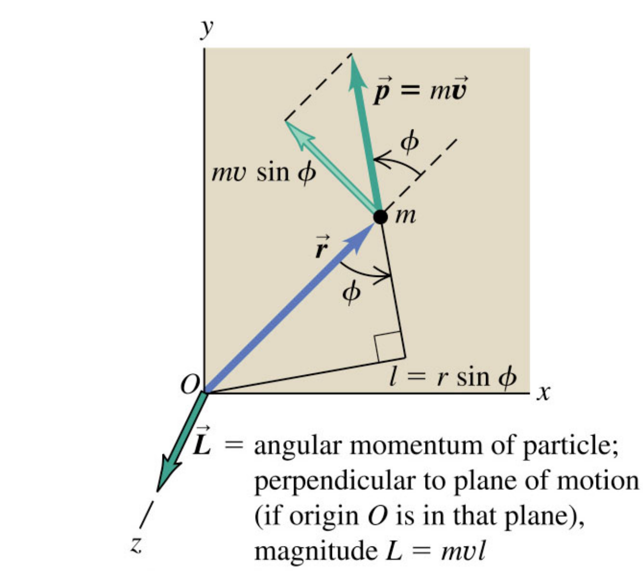
5 Angular Momentum

There is also a rotational analog to linear momentum; it's called *angular momentum.* The angular momentum of a mass or extended body depends on the choice of origin O.

$$\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}} = \vec{\mathbf{r}} \times m\vec{\mathbf{v}}$$

Notice the similarity between translational dynamics and rotational dynamics:

$$\vec{\tau} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$$
$$\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$$



Also notice that a particle **moving in a straight line** can also have angular momentum whose scalar value is defined as:

$$L = mvr\sin\phi = mv\ell$$

where ℓ is the **impact parameter**, similar to the *moment arm* we discussed when investigating the definition of torque.

We also have a rotational analog to Newton's 2nd law in terms of the angular momentum $\vec{\mathbf{L}}$:

$$\vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt} \qquad \rightarrow \qquad \vec{\tau} = \frac{d\vec{\mathbf{L}}}{dt}$$

The rate of change of angular momentum of a particle is the "sum of the torques" due to the forces acting on it.

$$L = \sum L_i = \sum m_i\, v_i\, r_i = \left(\sum m_i\, r_i^2\right)\omega = I\,\omega$$

where I is the moment of inertia about the z-axis. In vector form, we can write the angular momentum for an extended body rotating about around its symmetry axis as:

$$\vec{\mathbf{L}} = I\,\vec{\omega}$$

Ex. 38 A woman with mass 50 kg is standing on the rim of a large disk that is rotating at 0.80 rev/s about an axis through its center. The disk has mass 110 kg and a radius 4.0 m. Calculate the magnitude of the total angular momentum of the woman-plus-disk system. (Assume that you can treat the woman as a point.)

If multiple torques are applied to an extended body then it is proportional to the time-rate-of-change of the angular momentum:

$$\sum \vec{\tau} = \frac{d\vec{\mathbf{L}}}{dt}$$

6 Conservation of Angular Momentum

Similar to what we saw with **Newton's 3$^{\text{rd}}$ law** for translational motion:

$$\text{If} \quad \sum \vec{\mathbf{F}}_{\text{ext}} = 0 \qquad \text{then} \qquad \vec{\mathbf{p}} = \text{constant}$$

we also find that for rotational motion:

$$\text{If} \quad \sum \vec{\tau}_{\text{ext}} = 0 \qquad \text{then} \qquad \vec{\mathbf{L}} = \text{constant}$$

Ex. 43 Under some circumstances, a star can collapse into an extremely dense object made mostly of neutrons and called a *neutron star*. The density of a neutron star is roughly 10^{14} times as great as that of ordinary solid matter. Suppose we represent the star as a uniform, solid, rigid sphere, both before and after the collapse. The star's initial radius was 7.0×10^5 km (comparable to our sun); its final radius is 16 km. If the original star rotated once in 30 days, find the angular speed of the neutron star.

Ex. 48 **Asteroid Collision!** Suppose that an asteroid traveling straight toward the center of the earth were to collide with our planet at the equator and bury itself just below the surface. What would have to be the mass of this asteroid, in terms of the earth's mass M, for the day to become 25.0% longer than it presently is as a result of the collision? Assume that the asteroid is very small compared to the earth and that the earth is uniform throughout.

Pr. 71 **The Yo-yo** A yo-yo is made from two uniform disks each with mass m and radius R, connected by a light axle of radius b. A light, thin string is wound several times around the axle and then held stationary while the yo-yo is released from rest, dropping as the string unwinds. Find the linear acceleration and angular acceleration of the yo-yo and the tension in the string.

7 Gyroscopes and Precession

In this section we observe a peculiar property of objects that are *spinning*, and those that are *not spinning* while acted upon by some external force. Newton's 2[nd] law for rotational motional states:

$$\sum_i \vec{\tau}_i = \frac{d\vec{L}}{dt}$$

Another way of looking at Newton's 2[nd] law is to write it in *impulse* form:

$$d\vec{L} = \vec{\tau}\, dt \qquad \text{(Impulse form)}$$

7.1 Gyroscope spinning

Let's look at the case where the gyroscope **is spinning**.

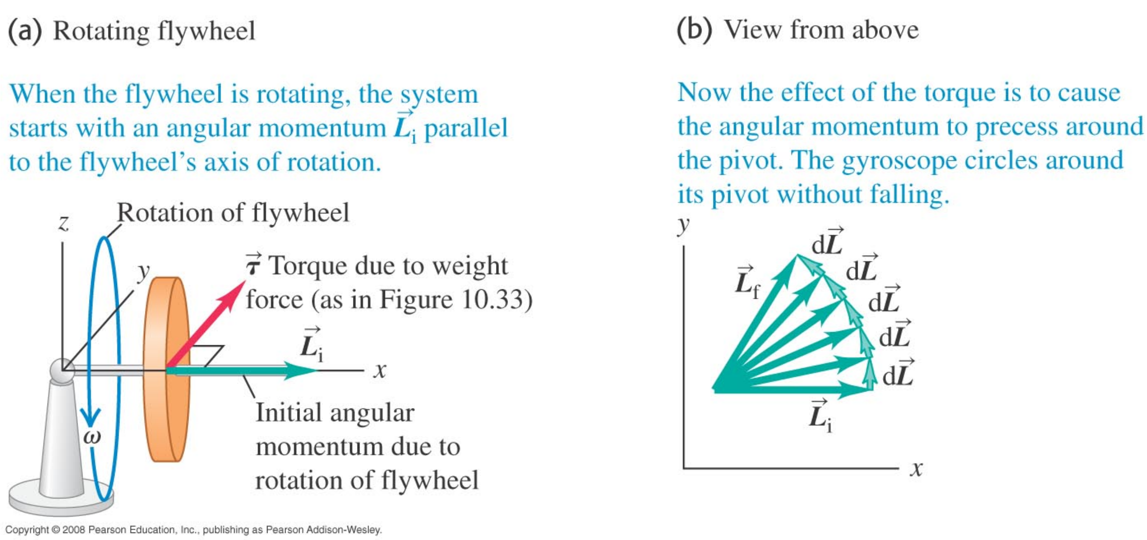


Figure 7: Figure 10.34 from University Physics 15[th] edition.

In the case of the spinning gyroscope we observe the axis of the gyroscope (or the angular momentum vector $\vec{L}$) rotating about the pivot point. This peculiar motion is called *precession*. The **precession angular speed** is denoted by the quantity Ω:

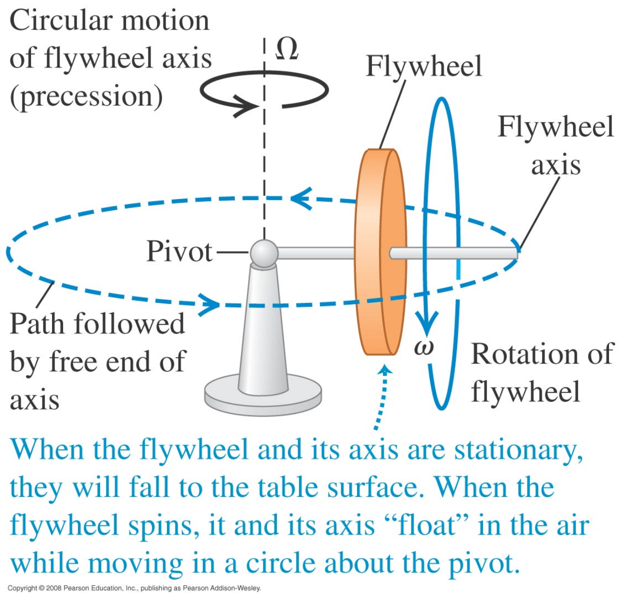


Figure 8: Figure 10.32 from University Physics 15$^{\text{th}}$ edition.

$$\Omega = \frac{d\phi}{dt} = \frac{|d\vec{L}|/|\vec{L}}{dt} = \frac{\tau_z}{L_z} = \frac{wr}{I\omega}$$

where w is the weight and r is the distance between the pivot and the gyroscope's center-of-mass when observed in the *top view*.

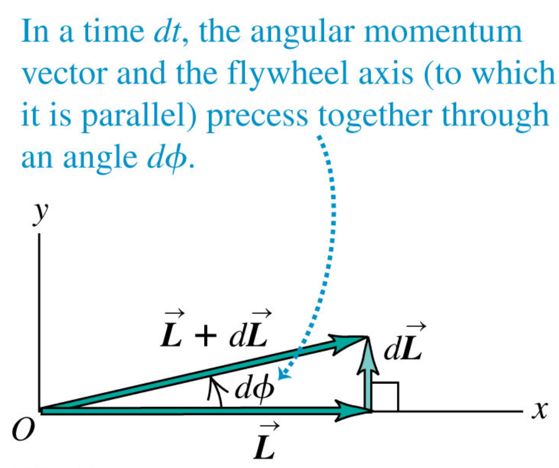


Figure 9: Figure 10.35 from University Physics 15$^{\text{th}}$ edition.

Ex. 56 **A gyroscope on the moon**. A certain gyroscope precesses at a rate of 0.50 rad/s when used on earth. If it were taken to a lunar base, where the acceleration due to gravity is 0.165 g, what would be its precession rate?

Prob. 85 A 500.0 g bird is flying horizontally at 2.25 m/s, not paying much attention, when it suddenly flies into a stationary vertical bar, hitting it 25.0 cm below the top (**Fig. P10.85**). The bar is uniform, 0.750 m long, has a mass of 1.50 kg and is hinged at its base. The collision stuns the bird so that it just drops to the ground afterward (but soon recovers to fly happily away). What is the angular velocity of the bar (a) just after it is hit by the bird and (b) just as it reaches the ground?

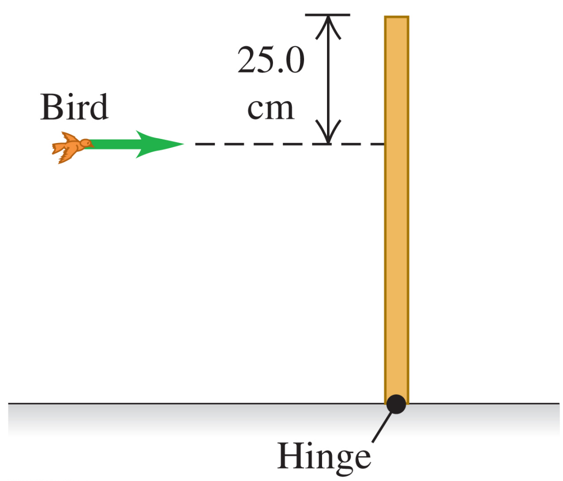


Figure 10: Figure P10.85 from University Physics 15^{th} edition.