Chapter 9
Rotation of Rigid Bodies

1 Angular Velocity and Acceleration

$$\theta = \frac{s}{r} \qquad \text{(angular displacement)}$$

The *natural* units of θ is radians.

$$1 \text{ rad} = \frac{360^o}{2\pi} = 57.3^o$$

Angular Velocity

Usually we pick the z-axis as the direction about which the rigid body rotates.

$$\bar{\omega} = \frac{\Delta\theta}{\Delta t} \qquad \text{(average angular velocity)}$$

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \qquad \text{(definition of angular velocity)}$$

The angular velocity can also be written as a vector. Its magnitude is defined by the above equation while its direction is defined by the *right-hand* rule.

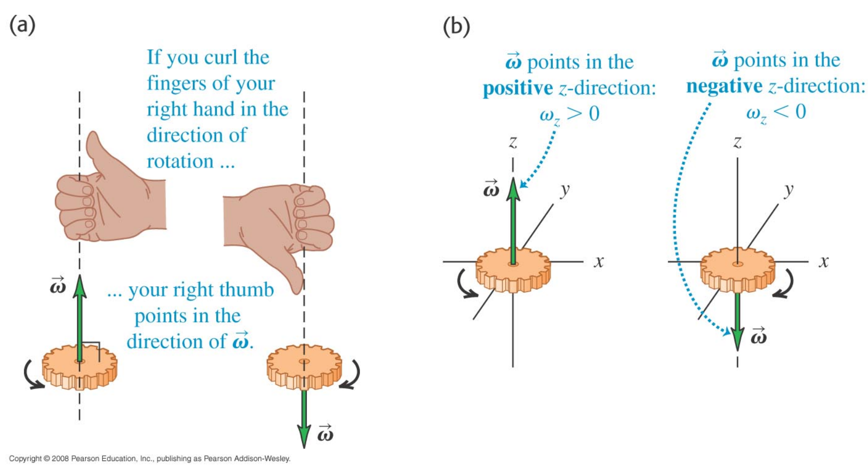

Figure 1: Figure 5 from Chapter 9 from University Physics 15$^{\text{th}}$ edition.

Angular Acceleration

The average angular acceleration is defined by:

$$\bar{\alpha} = \frac{\Delta\omega}{\Delta t} \qquad \text{(average angular acceleration)}$$

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} \qquad \text{(definition of angular acceleration)}$$

$$\omega = \frac{d\theta}{dt} \qquad \text{and} \qquad \alpha = \frac{d\omega}{dt} = \frac{d}{dt}\frac{d\theta}{dt} = \frac{d^2\theta}{dt^2}$$

Ex. 4 A fan blade rotates with angular velocity given by $\omega_z(t) = \gamma - \beta t^2$, where $\gamma = 5.00$ rad/s and $\beta = 0.800$ rad/s^3. a) Calculate the angular acceleration as a function of time. b) Calculate the instantaneous angular acceleration α_z at $t = 3.00$ s and the average angular acceleration $\alpha_{\text{av}-\text{z}}$ for the time interval $t = 0$ to $t = 3.00$ s. How do these two quantities compare? If they are different, why are they different?

1.1 Angular Acceleration As a Vector

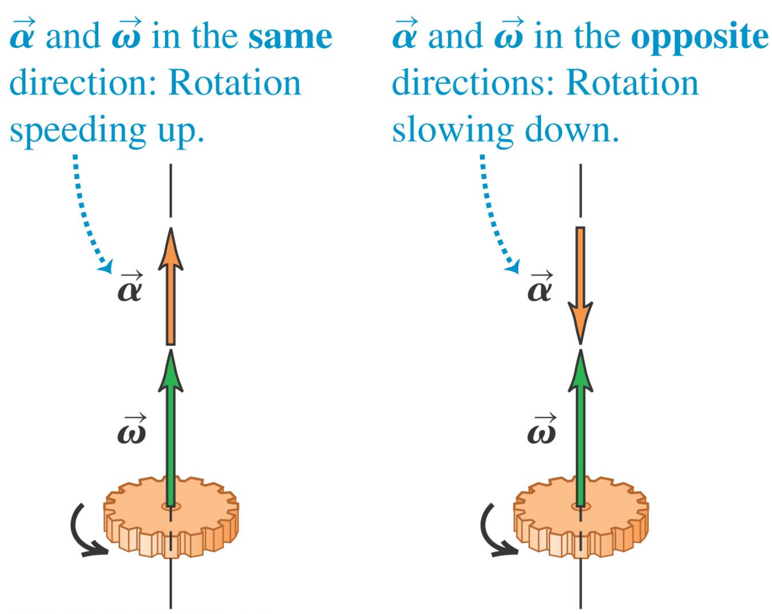

Figure 2: Figure 7 from Chapter 9 from University Physics 15$^{\text{th}}$ edition.

2 Rotation with Constant Angular Acceleration

$$\theta = \text{angular displacement}$$
$$\omega = \text{final angular velocity}$$
$$\omega_o = \text{initial angular velocity}$$
$$\alpha = \text{constant angular acceleration}$$
$$t = \text{time}$$

Here, we reintroduce the famous four equations for constant angular acceleration:

$$\omega = \omega_o + \alpha t \tag{1}$$
$$\theta = \frac{1}{2}\,(\omega + \omega_o)\,t \;=\; \bar{\omega}\,t \tag{2}$$
$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 \tag{3}$$
$$\omega^2 = \omega_o^2 + 2\alpha\theta \tag{4}$$

Ex. 14 A circular saw blade 0.200 m in diameter starts from rest. In 6.00 s it accelerates with constant angular acceleration to an angular velocity of 140 rad/s. Find the angular acceleration and the angle through which the blade has turned.

3 Relating Linear and Angular Kinematics

In this section, we introduce the relationship between *linear* and *angular* kinematical variables (for rigid bodies). Once we do this, we can use these relationships to determine the rotational dynamical quantities such as *rotational* kinetic energy (K_{rot}).

Starting with the relationship $s = r\theta$, we can calculate the *time rate of change* of both sides of this equation:

$$\left|\frac{ds}{dt}\right| = r\left|\frac{d\theta}{dt}\right|$$

or

$$v = r\omega \qquad \text{(relation between linear and angular speed)}$$

Likewise, if we look at the *time rate of change* of both sides of this equation, we find the following relation:

$$\frac{dv}{dt} = r\,\frac{d\omega}{dt} \qquad \text{or} \qquad a_{\tan} = r\,\alpha$$

There is a second acceleration, which we've seen before, namely a_{rad}.

$$a_{\text{rad}} = \frac{v^2}{r} = r\omega^2$$

Ex. 25 An advertisement claims that a centrifuge takes up only 0.127 m of bench space but can produce a radial acceleration of $3000g$ at 5000 rev/min. Calculate the required radius of the centrifuge. Is the claim realistic?

4 Energy in Rotational Motion

The kinetic energy of a point mass inside of a *rigid* body rotating about a *fixed* axis with angular velocity ω is:

$$\frac{1}{2}m_i v_i^2 = \frac{1}{2}m_i r_i^2\omega^2$$

where v_i is the tangential velocity of the point mass.

The *total* kinetic energy of a rigid body rotating about a fixed axis is:

$$K = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \cdots = \sum_{i=1}^{N}\frac{1}{2}\,m_i r_i^2\omega^2 = \frac{1}{2}\,\omega^2\sum_{i=1}^{N} m_i r_i^2 = \frac{1}{2}\,\omega^2\,I$$

where we define the *moment of inertia* (I):

$$I = \sum_{i=1}^{N} m_i r_i^2 \qquad \text{(definition of the moment of inertia)}$$

$$K = \frac{1}{2} I \omega^2 \qquad \text{(rotational kinetic energy of a rigid body)}$$

Let's look at the moments of inertia for various objects (Tablel 9.2).

Ex. 30 Four small spheres, each of which you can regard as a point of mass 0.200 kg, are arranged in a square 0.400 m on a side and connected by light rods (Fig. E9.30). Find the moment of inertia of the system about an axis a) through the center of the square, perpendicular to its plane (an axis through point O in the figure); b) bisecting two opposite sides of the sqaure (an axis along the line AB in the figure); c) that passes through the centers of the upper left and lower right spheres and through point O.

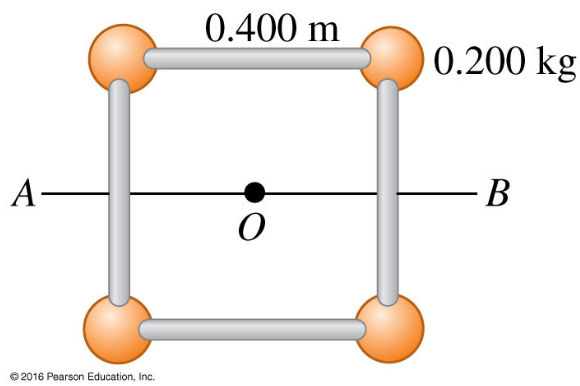

Figure 3: Fig. E9.30 from our textbook shows 4 point-masses forming a rigid body held together by 4 massless rods.

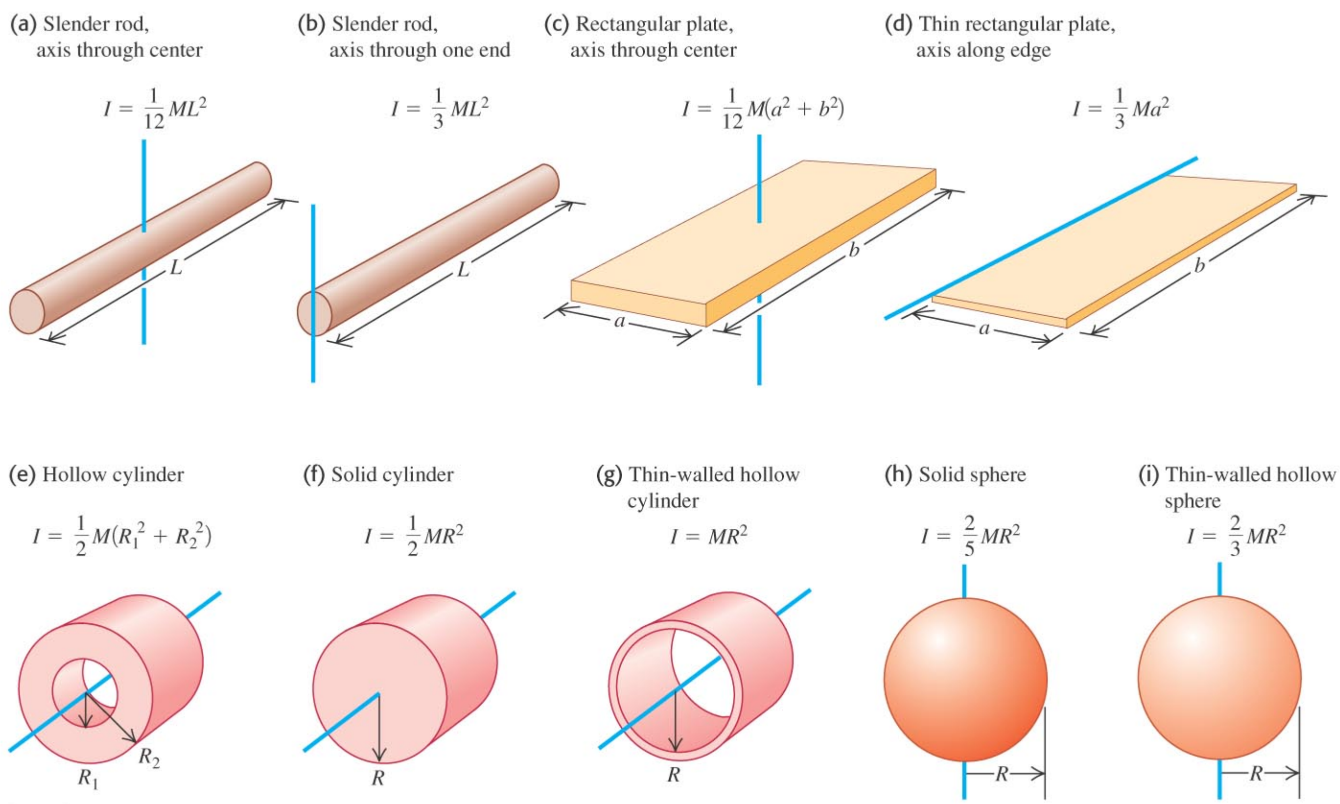

Figure 4: Moments of Inertia of Various Bodies

Ex. 47 A frictionless pulley has the shape of a uniform solid disk of mass 2.50 kg and radius 20.0 cm. A 1.50-kg stone is attached to a very light wire that is wrapped around the rim of the pulley (**Fig. E9.47**), and the system is released from rest. (a) How far must the stone fall so that the pulley has 4.50 J of kinetic energy? (b) What percent of the total kinetic energy does the pulley have?

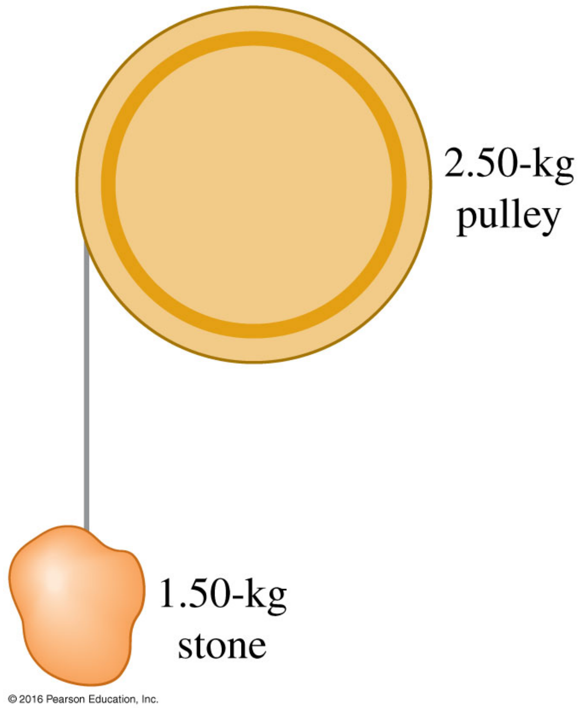

Figure 5: This is Fig. E9.47 from University Physics 15th edition.

4.1 Gravitational Potential Energy for an Extended Body

We can calculate the gravitational potential energy for an *extended* body by summing up the potential energies for all the masses (m_i) inside the body.

$$U_{\text{grav}} = m_1 g y_1 + m_2 g y_2 + m_3 g y_3 \cdots = g\,(m_1 y_1 + m_2 y_2 + m_3 y_3 + \cdots)$$

However, we know from our *center-of-mass* equations:

$$m_1 y_1 + m_2 y_2 + m_3 y_3 + \cdots = (m_1 + m_2 + m_3 + \cdots)\, y_{\text{cm}} = M\, y_{\text{cm}}$$

Thus,

$$U_{\text{grav}} = M g\, y_{\text{cm}} \qquad \text{(potential energy for an extended body)} \tag{5}$$

5 Parallel-Axis Theorem

If you know the moment of inertia of an object about an axis passing through its center-of-mass ($I_{\rm cm}$), then it's possible to *easily* calculate its new center-of-mass about another axis (that is parallel to the *original* axis). If the axis is translated a distance d, then the new moment of inertia can be written as:

$$I_{\rm new} = I_{\rm cm} + M\,d^2 \qquad \text{(parallel-axis theorem)} \tag{6}$$

In many cases, calculating the moment of inertia requires doing an integral $I = \int r^2\,dm$. Calculating the moment of inertia for an object rotating around a *new* fixed axis (parallel to the original axis) would normally require doing another integral $I_{\rm new} = \int r^2\,dm$. The parallel axis theorem states, that if you know the moment of inertia about the center-of-mass ($I_{\rm cm}$), then the new moment of inertia for an axis parallel to the original axis can be easily determined without doing any integration.

Let's look at an example using the $I_{\rm cm}$'s shown in Table 9.2.

Ex. 53 A thin, rectangular sheet of metal has mass M and sides of length a and b. Use the parallel-axis theorem to calculate the moment of inertia of the sheet for an axis that is perpendicular to the plane of the sheet and that passes through one corner of the sheet.

Ex. 56 Use $I = \int r^2\,dm$ to calculate the moment of inertia of a slender, uniform rod with mass M and length L about an axis at one end perpendicular to the rod.

Prob. 77 The pulley in **Fig. P9.77** has radius R and a moment of inertia I. The rope does not slip over the pulley, and the pulley spins on a frictionless axle. The coefficient of kinetic friction between block A and the table top is μ_k. The system is released from rest, and block B descends. Block A has a mass m_A and block B has mass m_B. Use energy methods to calculate the speed of block B as a function of the distance d that it has descended.

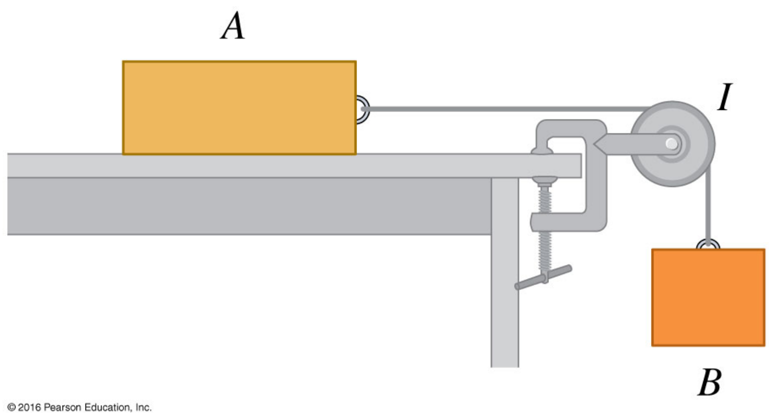

Figure 6: This is Fig. P9.77 from University Physics 15$^{\text{th}}$ edition.

Prob. 86 **Neutron Stars and Supernova Remnants.** The Crab Nebula is a could of glowing gas about 10 light-years across, located about 6500 light-years from the earth **(Fig. P9.86)**. It is the remnant of a star that underwent a *supernova explosion*, seen on earth in 1054 A.D. Energy is release by the Crab Nebula at a rate of about 5×10^{31} W, about 10^5 times the rate at which the sun radiates energy. The crab Nebula obtains its energy from the rotational kinetic energy of a rapidly spinning *neutron star* at its center. The object rotates once every 0.0331 s, and this period is increasing by 4.22×10^{-13} s for each second of time that elapses. (a) If the rate at which energy is lost by the neutron star is equal to the rate at which energy is released by the nebula, find the moment of inertia of the neutron star. (b) Theories of supernovae predict that the neutron star in the Crab Nebula has a mass about 1.4 times that of the sun. Modeling the neutron star as a solid uniform sphere, calculate its radius in kilometers. (c) What is the linear speed of a point on the equator of the neutron star? Compare it to the speed of light. (d) Assume that the neutron star is uniform and calculate its density. Compare it to the density of ordinary rock (3000 kg/m^3) and to the density of an atomic nucleus (about 10^{17} kg/m^3). Justify the statement that a neutron star is essentially a large atomic nucleus.

Figure 7: P9.86 from University Physics 15$^{\text{th}}$ edition.