Chapter 8
Momentum and Impulse

Momentum plays a pivotal role in extending our understanding of Newton's Laws. In fact, Newton's laws were first written in terms of momentum. Later in this chapter, we will discover that, like energy, momentum is also a conserved quantity in our universe.

1 Momentum and Impulse

When we previously studied Newton's 2^{nd} law, we wrote that:

$$\sum \vec{F} = m\vec{a}$$

However, if we write it in the following way (similar to what Newton did), we find:

$$\sum \vec{F} = m\frac{d\vec{v}}{dt} = \frac{d}{dt}(m\vec{v})$$

thus, the sum of the forces is equal to the time rate of $m\vec{v}$, which we call the **momentum**, or **linear momentum**.

$$\vec{\mathbf{p}} = m\vec{\mathbf{v}} \qquad \text{(definition of momentum)} \tag{1}$$

Momentum ($\vec{\mathbf{p}}$) is a vector quantity with SI units of kg·m/s.

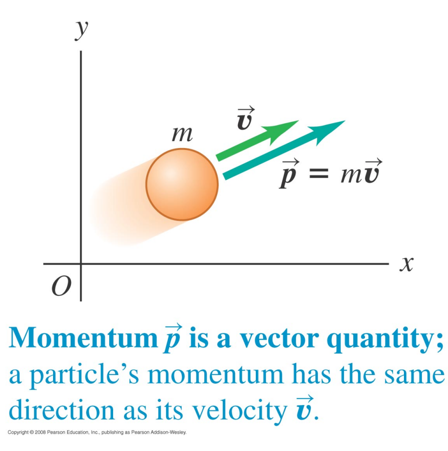


Figure 1: Figure 8.1 from University Physics 12^{th} edition.

If we write Newton's second law in terms of momentum, we find the following:

$$\sum \vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt} \tag{2}$$

The advantage to writing Newton second law this way is that the $\sum \vec{\mathbf{F}}$ is related to both the change in velocity vector ($\vec{\mathbf{v}}$) and the change in mass m.

Using our definition of momentum Eq. 1, we decompose this vector equation into three scalar equations:

$$p_x = m\,v_x \qquad p_y = m\,v_y \qquad p_z = m\,v_z \tag{3}$$

It is more instructive to look at Eq. 1 when applying Newton's second law to a system where you want to investigate the motion of an object in all three dimensions at the same time. However, in solving most problems, we will more frequently use the scalar definitions described in Eq. 3. This is especially true once we've established the principle of *conservation of momentum*.

Ex. 4 Two vehicles are approaching an intersection. One is a 2500-kg pickup traveling at 14.0 m/s from east to west (the -x-direction), and the other is a 1500-kg sedan going from south to north (the +y-direction) at 23.0 m/s. (a) Find the x- and y-components of the net momentum of this system. (b) What are the magnitude and direction of the net momentum?

Let's consider a particle that is acted upon by a *constant* net force $\sum \vec{\mathbf{F}}$ during a time interval Δt. We define the impulse to be the product of the net force and the time interval:

$$\vec{\mathbf{J}} = \sum \vec{\mathbf{F}}\,(t_2 - t_1) = \sum \vec{\mathbf{F}}\,\Delta t \qquad \text{(assuming constant net force)} \tag{4}$$

The direction of the impulse is in the same direction as $\sum \vec{\mathbf{F}}$, and has SI units of kg·m/s, the same as linear momentum. In fact, this suggests that there might be a connection between *impulse* and *linear momentum*. Let's rewrite Newton's second law in the following manner:

$$\sum \vec{\mathbf{F}} = \frac{\Delta\vec{\mathbf{p}}}{\Delta t} = \frac{\vec{\mathbf{p}}_2 - \vec{\mathbf{p}}_1}{t_2 - t_1}$$

Then, rearranging terms we have:

$$\sum \vec{\mathbf{F}}\,(t_2 - t_1) = \vec{\mathbf{p}}_2 - \vec{\mathbf{p}}_1$$

Using our definition of *impulse* from Eq. 4, we arrive at the **impulse-momentum theorem**:

$$\vec{\mathbf{J}} = \vec{\mathbf{p}}_2 - \vec{\mathbf{p}}_1 \qquad \text{(impulse-momentum theorem)} \tag{5}$$

The change in momentum of a particle equals the net force multiplied by the time interval over which the net force is applied.

If the $\sum \vec{\mathbf{F}}$ is not constant, we can integrate both sides of Newton's second law $\sum \vec{\mathbf{F}} = d\vec{\mathbf{p}}/dt$ over the time interval $t_1 \to t_2$:

$$\int_{t_1}^{t_2} \sum \vec{\mathbf{F}}\, dt = \int_{t_1}^{t_2} \frac{d\vec{\mathbf{p}}}{dt}\, dt = \int_{\vec{\mathbf{p}}_1}^{\vec{\mathbf{p}}_2} d\vec{\mathbf{p}} = \vec{\mathbf{p}}_2 - \vec{\mathbf{p}}_1$$

where the integral on the left-hand side is defined to be the impulse $\vec{\mathbf{J}}$ due to the net force $\sum \vec{\mathbf{F}}$ over the time interval $(t_1 \to t_2)$:

$$\vec{\mathbf{J}} = \int_{t_1}^{t_2} \sum \vec{\mathbf{F}}\, dt \qquad \text{(general definition of impulse)}$$

We can define an *average* net force $\vec{\mathbf{F}}_{\text{av}}$ such that even when $\sum \vec{\mathbf{F}}$ is not constant, the impulse $\vec{\mathbf{J}}$ is given by

$$\vec{\mathbf{J}} = \vec{\mathbf{F}}_{\text{av}}\,(t_2 - t_1) \tag{6}$$

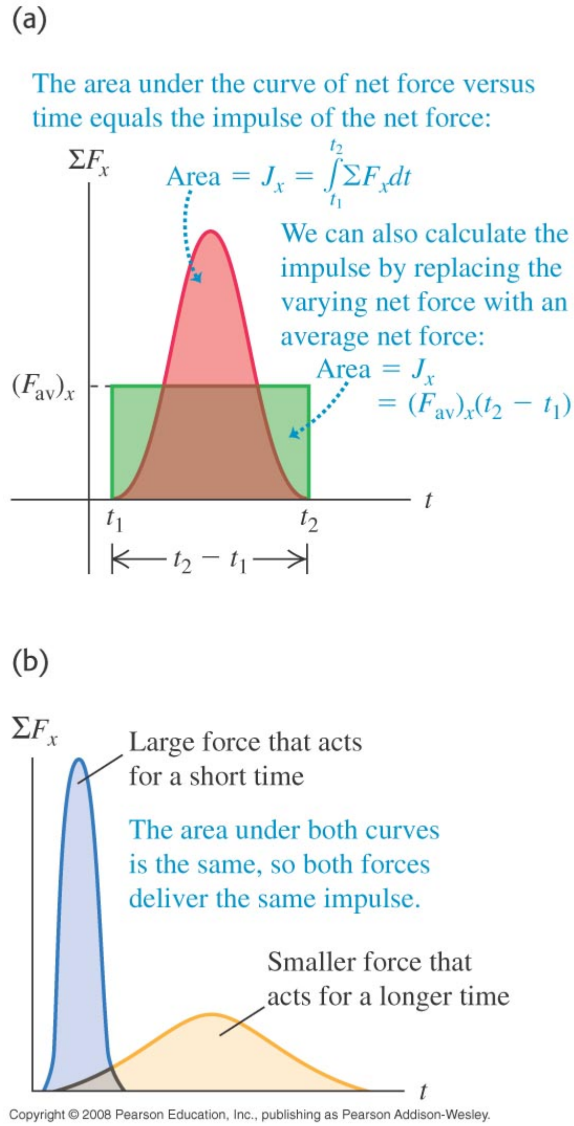


Figure 2: Figure 8.3 from University Physics 12[th] edition

Ex. 8 Force of a Baseball Swing. A baseball has a mass 0.145 kg. a) If the velocity of a pitched ball has a magnitude of 45.0 m/s and the batted ball's velocity is 55.0 m/s in the opposite direction, find the magnitude of the change in momentum of the ball and of the impulse applied to it by the bat. b) If the ball remains in contact with the bat for 2.00 ms, find the magnitude of the average force applied by the bat.

1.1 Momentum and Kinetic Energy Compared

When a net force $\sum \vec{\mathbf{F}}$ is applied to a system, the kinetic energy and the momentum undergo change according to the following equations:

$$\Delta\vec{\mathbf{p}} = \vec{\mathbf{p}}_2 - \vec{\mathbf{p}}_1 = \left(\sum \vec{\mathbf{F}}\right) \Delta t$$

$$\Delta K = K_2 - K_1 = \left(\sum \vec{\mathbf{F}}\right) \cdot \vec{\mathbf{s}}$$

1.2 Relationship Between Kinetic Energy and Momentum

As you can see from the above equation, the $\sum \vec{F}$ (the *net* force) forms a relationship between the *change* in momentum and the *change* in kinetic energy. The relationship between the *kinetic energy* ($\frac{1}{2}mv^2$) and the *momentum* (mv) is:

$$K = \frac{p^2}{2m} \qquad \text{or} \qquad K = \frac{\vec{p} \cdot \vec{p}}{2m} \tag{7}$$

$$dK = \frac{1}{2m}\left(\vec{p} \cdot d\vec{p} + d\vec{p} \cdot \vec{p}\right) = \frac{1}{2m}\left(2\vec{p} \cdot d\vec{p}\right) = \frac{\vec{p}}{m} \cdot d\vec{p} = \vec{v} \cdot d\vec{p}$$

This relationship hold when $dp \ll p$.

2 Conservation of Momentum

When two bodies A and B interact with each other but not with anything else, the forces exerted on each body must be described by Newton's *third* law, namely, the two forces are always equal in magnitude and opposite in direction. Furthermore, the forces are internal forces, so the sum of the forces must be equal to zero.

$$\vec{\mathbf{F}}_{\text{B on A}} + \vec{\mathbf{F}}_{\text{A on B}} = \vec{\mathbf{0}}$$

$$\vec{\mathbf{F}}_{\text{B on A}} + \vec{\mathbf{F}}_{\text{A on B}} = \frac{d\vec{\mathbf{p}}_{\text{A}}}{dt} + \frac{d\vec{\mathbf{p}}_{\text{B}}}{dt} = \frac{d}{dt}\left(\vec{\mathbf{p}}_A + \vec{\mathbf{p}}_B\right) = \vec{\mathbf{0}} \tag{8}$$

Now we define the total momentum of the two-body system to be:

$$\vec{\mathbf{P}} = \vec{\mathbf{p}}_A + \vec{\mathbf{p}}_B \qquad \text{(the total momentum)}$$

Substituting our equation for *total momentum* into Eq. 8, we find:

$$\vec{\mathbf{F}}_{\text{B on A}} + \vec{\mathbf{F}}_{\text{A on B}} = \frac{d\vec{\mathbf{P}}}{dt} = \vec{\mathbf{0}}$$

If the vector sum of the external forces on a system is zero, the total momentum of the system is constant.

This is the **principle of conservation of momentum.** The total momentum can be expanded to N particles:

$$\vec{\mathbf{P}} = \vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2 + \vec{\mathbf{p}}_3 + \cdots = \sum_{i=1}^{N} \vec{\mathbf{p}}_i \qquad \text{(the total momentum)}$$

Again, this is a vector equation that really represents three scalar equations:

$$P_x = p_{1x} + p_{2x} + p_{3x} + \cdots = m_1\, v_{1x} + m_2\, v_{2x} + m_3\, v_{3x} + \cdots$$
$$P_y = p_{1y} + p_{2y} + p_{3y} + \cdots = m_1\, v_{1y} + m_2\, v_{2y} + m_3\, v_{3y} + \cdots$$
$$P_z = p_{1z} + p_{2z} + p_{3z} + \cdots = m_1\, v_{1z} + m_2\, v_{2z} + m_3\, v_{3z} + \cdots$$

N.B. Momentum is separately conserved in the x, y, and z directions.

Ex. 16 A 68.5-kg astronaut is doing a repair in space on the orbiting space station. She throws a 2.25-kg tool away from her at 3.20 m/s relative to the space station. With what speed and in what direction will she begin to move?

Ex. 32 Two skaters collide and grab on to each other on frictionless ice. One of them, of mass 70.0 kg, is moving to the right at 4.00 m/s, while the other, of mass 65.0 kg, is moving to the left at 2.50 m/s. What are the magnitude and direction of the velocity of these skaters just after they collide?

Ex. 30 **Asteroid Collision.** Two asteroids of equal mass in the asteroid belt between Mars and Jupiter collide with a glancing blow. Asteroid A, which was initially traveling at 40.0 m/s, is deflected 30.0° from its original direction, while asteroid B which was initially at rest, travels at 45.0° to the original direction of A (Fig. E8.30). (a) Find the speed of each asteroid after the collision. (b) What fraction of the original kinetic energy of asteroid A dissipates during this collision?

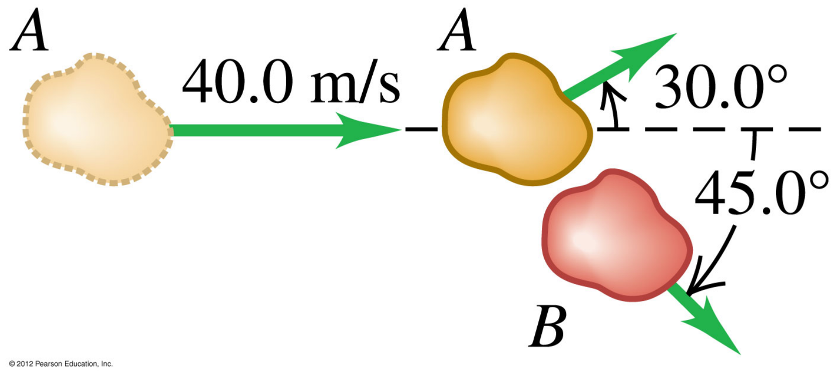


Figure 3: Figure 8.30 from University Physics 15th edition.

3 Inelastic Collisions

First of all, we need to define what constitutes a collision. When we think of collisions, we naturally conjure up a picture of two objects coming into contact with each other, possibly doing some damage to each other, and possibly fragmenting off bits and pieces in the process. However, this describes only one class of collisions, *inelastic* collisions. There is another class of collisions called *elastic* collisions where physical contact may or may not occur. We will describe elastic collisions in the next section.

Inelastic collisions come in two types: inelastic, and *completely* inelastic collisions. If two objects collide and "stick" together and move off together as "one" object, then this is called an **inelastic collision**.

Let's look at a collision between two particle A and B whose initial and final motions are confined to the x direction. Let's label the initial velocities with the subscript 1 and the final velocities with the subscript 2. **In an inelastic collision, momentum is the only quantity conserved**.

$$
\begin{aligned}
m_A v_{A1x} + m_B v_{B1x} &= m_A v_{A2x} + m_B v_{B2x} \qquad \text{(for all collisions)} \\
m_A v_{A1x} + m_B v_{B1x} &= (m_A + m_B)\, v_f \qquad \text{(for a completely inelastic collision)}
\end{aligned}
$$

Ex. 43 **A Ballistic Pendulum.** A 12.0-g rifle bullet is fired with a speed of 380 m/s into a ballistic pendulum with mass 6.00 kg, suspended from a cord 70.0 cm long (see Example 8.8 in Section 8.3). Compute a) the vertical height through which the pendulum rises; b) the initial kinetic energy of the bullet; c) the kinetic energy of the bullet and pendulum immediately after the bullet becomes embedded in the pendulum.

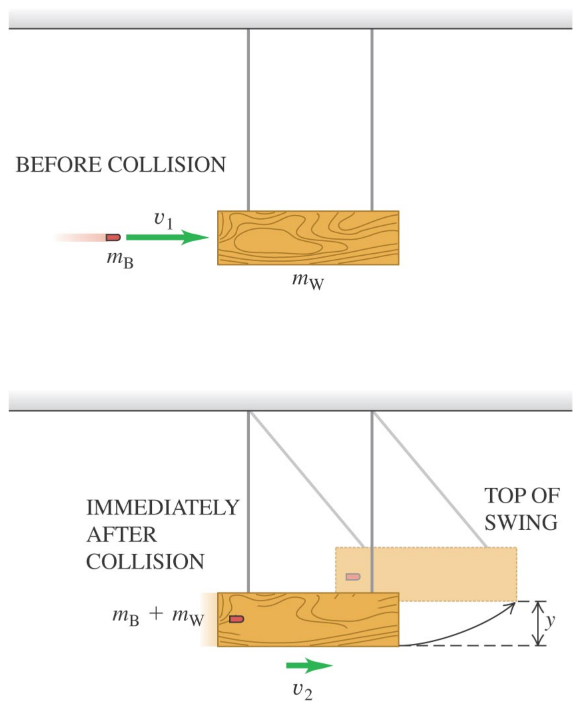


Figure 4: Figure 8.19 from University Physics 15th edition.

4 Elastic Collisions

In an **elastic collision**, the momentum and kinetic energy are both conserved.

4.1 Elastic Collisions in One Dimension

$$
m_A v_{Ai} + m_B v_{Bi} = m_A v_{Af} + m_B v_{Bf} \qquad \text{(for all collisions)}
$$

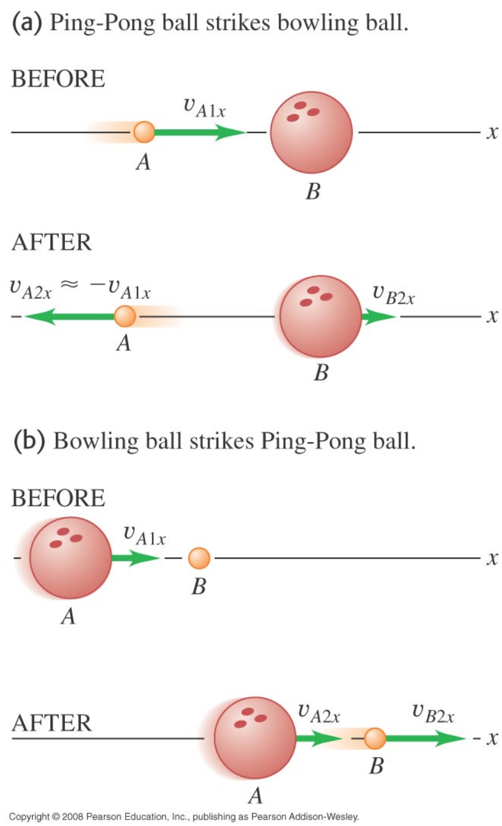


Figure 5: Figure 8.22 from University Physics 12 edition.

$$\frac{1}{2}m_A v_{Ai}^2 + \frac{1}{2}m_B v_{Bi}^2 = \frac{1}{2}m_A v_{Af}^2 + \frac{1}{2}m_B v_{Bf}^2 \qquad \text{(Conservation of KE)}$$

Let's imagine we have a collision in one dimension (x-axis) between objects A and B, and the resulting motion after the collision is also along the x axis. We will let the subscript "i" denote the velocities before the collision and the subscript "f" denote the velocities after the collision. In this situation, we have both *conservation of kinetic energy* and *conservation of momentum*. By using these two conservation laws and knowing the initial velocities, we can find the final velocities to be the following :

In one dimension:

$$v_{Af} = \frac{m_A - m_B}{m_A + m_B} v_{Ai} + \frac{2m_B}{m_A + m_B} v_{Bi}$$

$$v_{Bf} = \frac{2m_A}{m_A + m_B} v_{Ai} + \frac{m_B - m_A}{m_A + m_B} v_{Bi}$$

Ex. 50 You are at the controls of a particle accelerator, sending a beam of 1.50×10^7 m/s protons (mass m) at a gas target of an unknown element. Your detector tells you that some protons bounce straight back after a collision with one of the nuclei of the unknown element. All such protons rebound with a speed of 1.20×10^7 m/s. Assume that the initial speed of the target nucleus is negligible and that the collision is elastic. a) Find the mass of one nucleus of the unknown element. Express your answer in terms of the proton mass m. b) What is the speed of the unknown nucleus immediately after such a collision?

4.2 Two-dimensional elastic collision

See Example 8.12 in our textbook. Three equations and three unknowns.

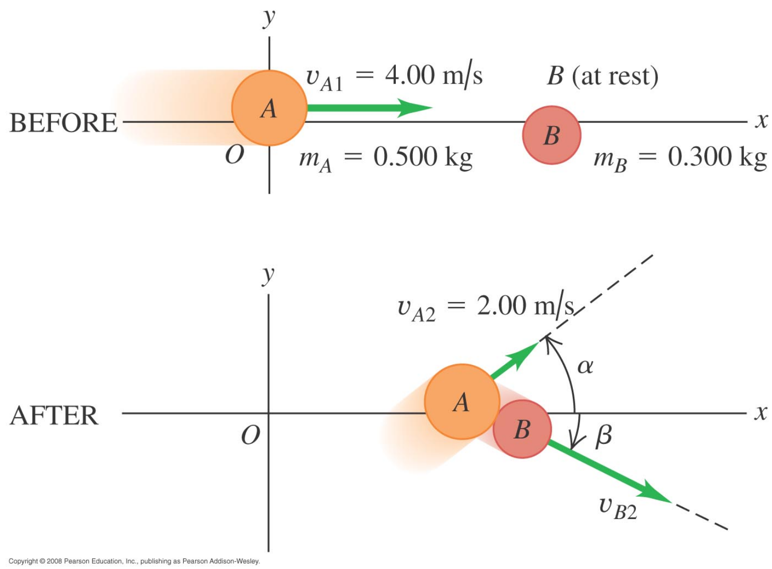


Figure 6: Figure 8.26 from University Physics 13th edition.

5 Center of Mass

$$x_{\text{cm}} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \cdots}{m_1 + m_2 + m_3 + \cdots}$$

$$y_{\text{cm}} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3 + \cdots}{m_1 + m_2 + m_3 + \cdots}$$

$$z_{\text{cm}} = \frac{m_1 z_1 + m_2 z_2 + m_3 z_3 + \cdots}{m_1 + m_2 + m_3 + \cdots}$$

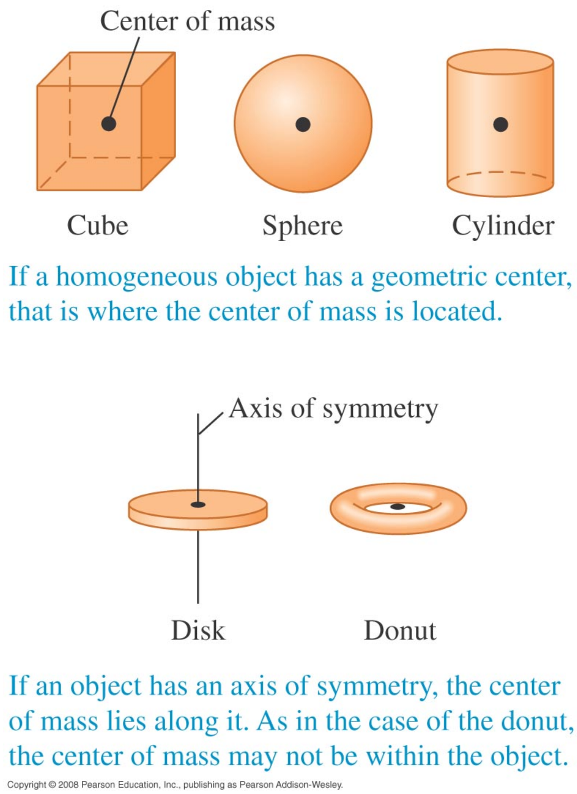


Figure 7: Figure 8.28 froom University Physics 12^{th} edition.

Ex. 52 Find the position of the center of mass of the system of the sun and Jupiter. (Since Jupiter is more massive than the rest of the planets put together, this is essentially the position of the center of mass of the solar system.) Does the center of mass lie inside or outside the sun? Use the data in Appendix F. ($\mathrm{M_{Sun}} = 1.99 \times 10^{30}$ kg, and $\mathrm{M_{Jupiter}} = 1.90 \times 10^{27}$ kg, and $d_{\text{jupiter}-\text{sun}} = 7.78 \times 10^{11}$ m)

5.1 Motion of the Center of Mass

$$V_{\text{cm}-\text{x}} = \frac{m_1 v_{1x} + m_2 v_{2x} + m_3 v_{3x} + \cdots}{m_1 + m_2 + m_3 + \cdots}$$

$$V_{\text{cm}-\text{y}} = \frac{m_1 v_{1y} + m_2 v_{2y} + m_3 v_{3y} + \cdots}{m_1 + m_2 + m_3 + \cdots}$$

$$V_{\text{cm}-\text{z}} = \frac{m_1 v_{1z} + m_2 v_{2z} + m_3 v_{3z} + \cdots}{m_1 + m_2 + m_3 + \cdots}$$

$$\vec{\mathbf{V}}_{\mathrm{cm}} = \frac{m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2 + m_3\vec{\mathbf{v}}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\vec{\mathbf{P}}}{M} \tag{9}$$

The total momentum can be written as:

$$\vec{\mathbf{P}} = M\,\vec{\mathbf{V}}_{\mathrm{cm}}$$

5.2 External Forces and Center-of-Mass Motion

If the net external force on a system of particles is not zero, then the total momentum is not conserved and the velocity of the center-of-mass ($\vec{v}_{cm}$) changes. Because of Newton's third law, the internal forces all cancel in pairs, and $\sum \vec{F}_{int} = \vec{0}$. What survives on the left-hand side of Newton's second law is the sum of the *external* forces:

$$\sum \vec{F}_{ext} = M\vec{a} \qquad \text{(where } M \text{ is the mass of the body or collection of particles)}$$

When a body of a collection of particles is acted on by external forces, the center-of-mass moves just as though all the mass were concentrated at that point and it were acted on by a net force equal to the sum of the external forces on the system.

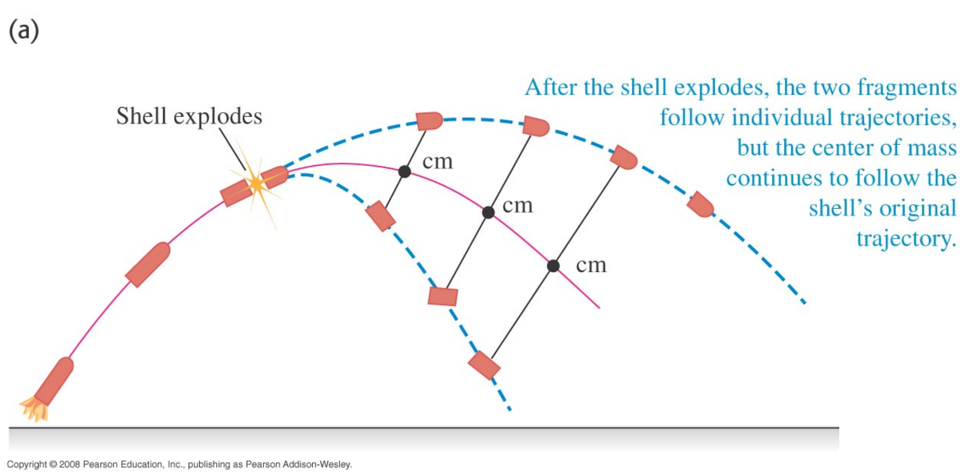


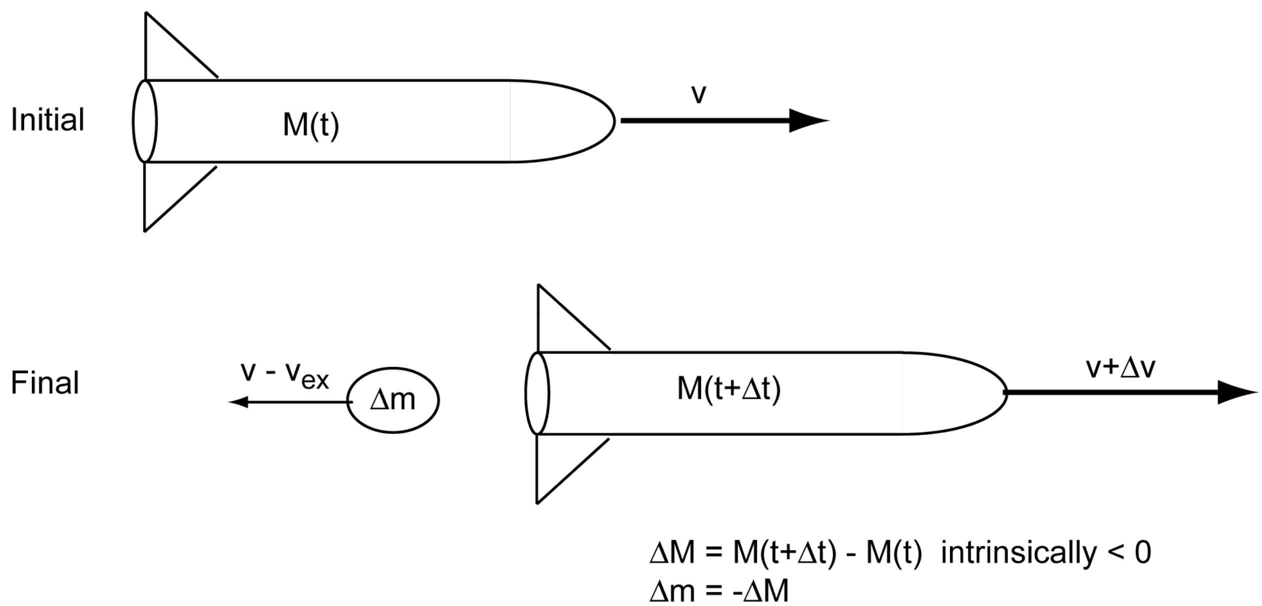
Figure 8: Figure 8.31a from University Physics 13$^{\text{th}}$ edition.

$$\sum \vec{F}_{ext} = M\vec{a}_{cm} = \frac{d\vec{P}}{dt}$$

Note:
If the net external force is zero, then the acceleration of the center-of-mass is zero (i.e., $\vec{a}_{cm} = 0$). Likewise, if the net external force is zero, then the total momentum $\vec{P}$ must be *conserved*, (i.e., $d\vec{P}/dt = 0$).

6 Rocket Propulsion

Using conservation of momentum, we can write the following:



Momentum Initial = Momentum Final

$$M(t)\,v = M(t+\Delta t)\,(v+\Delta v) + \Delta m(v - v_{\rm ex})$$

$$0 = (M(t+\Delta t) - M(t))\,v + M(t+\Delta t)\Delta v + v\,\Delta m - v_{\rm ex}\Delta m$$

However, $M(t+\Delta t) - M(t) = \Delta M = -\Delta m$, so, the *first* and *third* terms cancel and we can rewrite the previous equation as:

$$0 = M(t+\Delta t)\Delta v - v_{\rm ex}\Delta m \tag{10}$$

At this point, we can go in two different directions with equation 10. We can develop the *thrust* equation, or the *velocity* equation. Let's start with the *thrust* equation.

The Thrust Equation
We can make the substitution in Eq. 10 where we divide everything by Δt, and take the limit as $\Delta t \to 0$. Then, we have:

$$0 \;=\; M(t)\,\frac{dv}{dt} - v_{\rm ex}\,\frac{dm}{dt} \qquad \text{or} \qquad M\frac{dv}{dt} = v_{\rm ex}\,\frac{dm}{dt}$$

$$\text{Force on the rocket} \;=\; v_{\rm ex}\frac{dm}{dt} \;=\; Ma \qquad \text{where} \qquad \text{Thrust} \;=\; v_{\rm ex}\frac{dm}{dt} \tag{11}$$

The reason that my "sign" convention is different from the book (thrust $= -v_{\rm ex}dm/dt$) is because I chose $dm > 0$, while the book chose $dm < 0$.

Equation 10 describes Newton's 2$^{\rm nd}$ law for a rocket in space *without* any other external forces. If the rocket is situated on a launch pad, then you would have to add the gravitational pull of the earth. Near the earth's surface, we would have the following equation:

$$\text{Force on the rocket} = v_{\rm ex}\,\frac{dm}{dt} \;-\; Mg \;=\; Ma \tag{12}$$

The Velocity Equation

Starting with Eq. 10, we make the following substitution: $\Delta m \to -\Delta M$, and take the *limit* as $\Delta t \to 0$, and gathering the "mass" terms on one side of the equation we have:

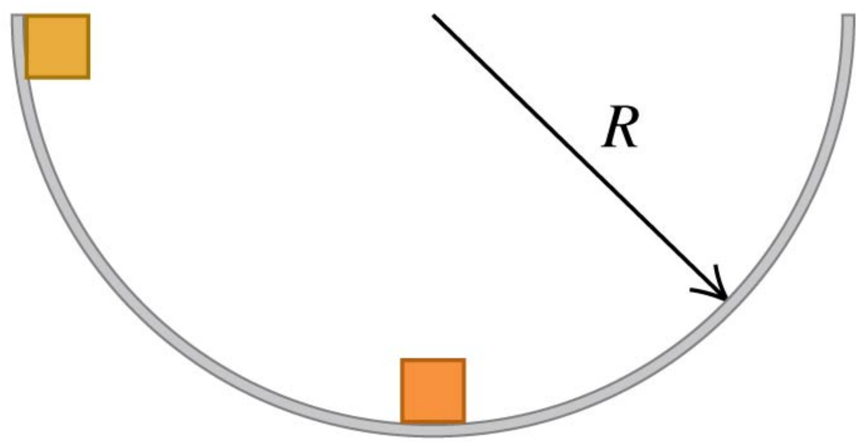
$$dv \;=\; -v_{\rm ex}\frac{dM}{M} \qquad\qquad \Rightarrow \qquad\qquad \int_{v_o}^{v} dv \;=\; -v_{\rm ex}\int_{M_{\rm i}}^{M_{\rm f}}\frac{dM}{M}$$

$$v - v_o \;=\; v_{\rm ex}\,\ell n\left(\frac{m_o}{m}\right) \qquad \text{(the velocity equation)}$$

Ex. 61 A 70-kg astronaut floating in space in a 110-kg MMU (manned maneuvering unit) experiences an acceleration of 0.029 m/s^2 when he fires one of the MMU's thrusters. a) If the speed of the escaping N_2 gas relative to the astronaut is 490 m/s, how much gas is used by the thruster in 5.0 s? b) What is the thrust of the thruster?

Other Homework Problems

Ex. 82 Two identical masses are released from rest in a smooth hemispherical bowl of radius R from the positions shown in **Fig. P8.82.** Ignore friction between the masses and the surface of the bowl. If the masses stick together when they collide, how high above the bottom of the bowl will they go after colliding?

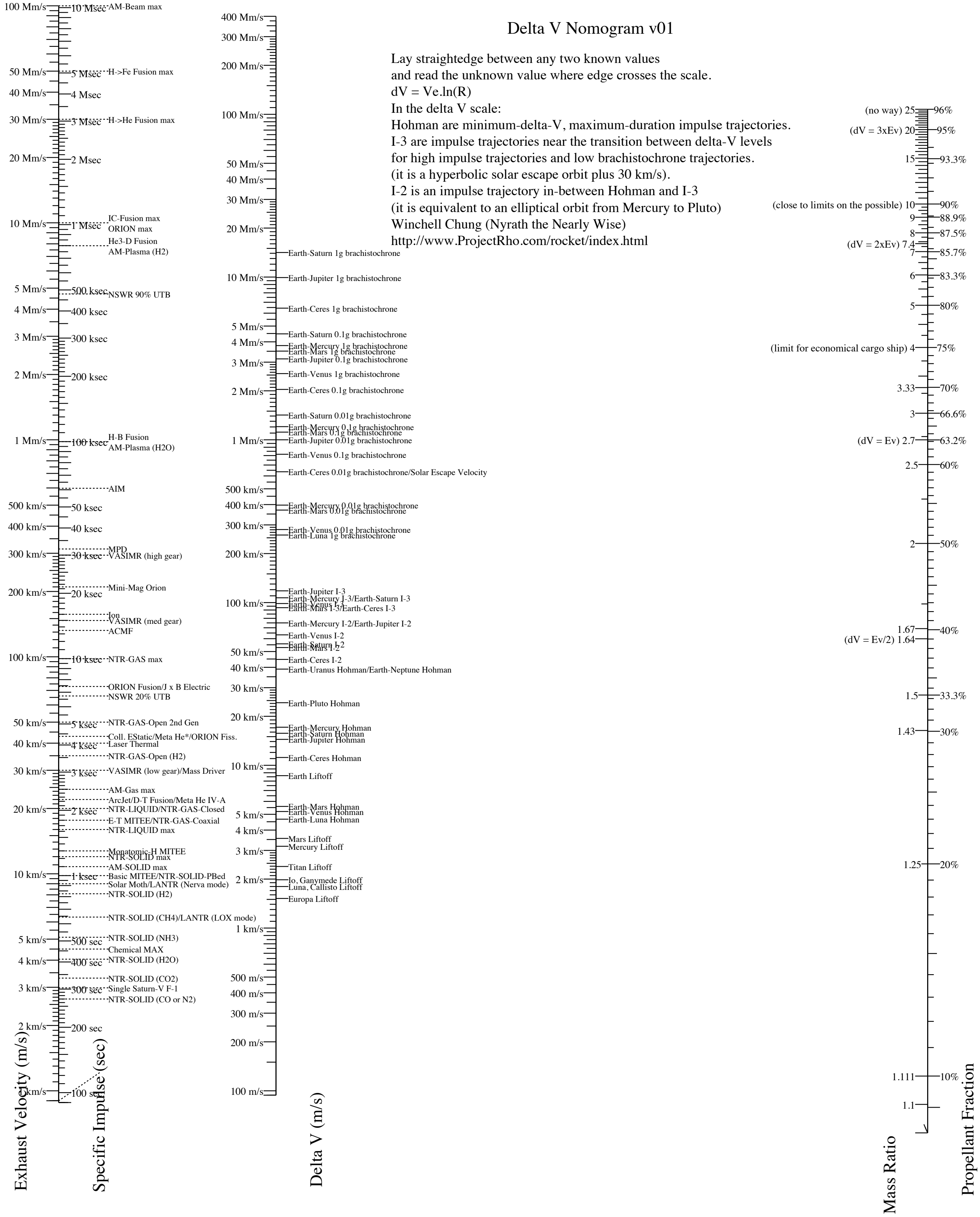


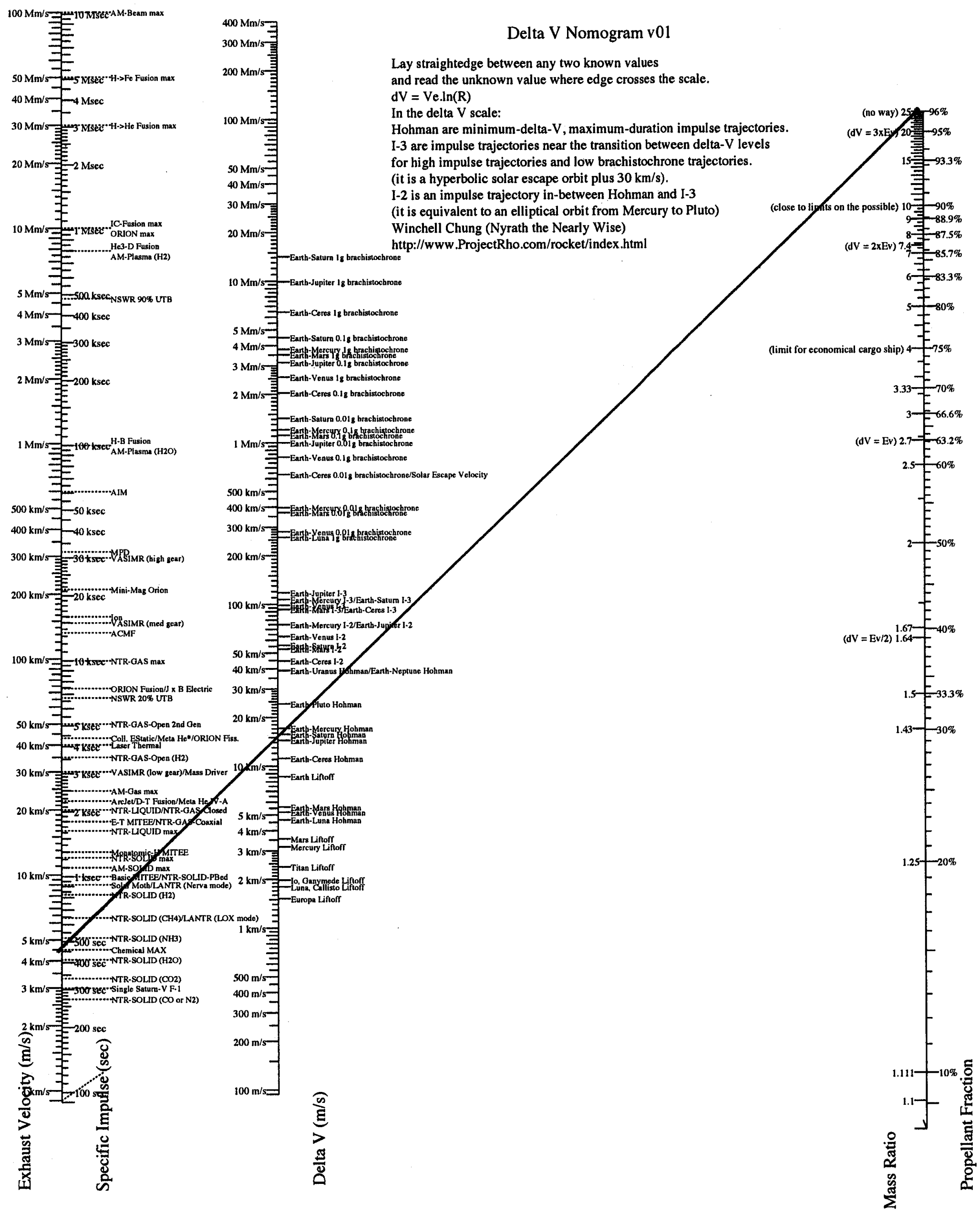
Exercise A proton moving with speed v_{A1} in the $+x$-direction makes an elastic, off-center collision with an identical proton originally at rest. After impact, the first proton moves with speed v_{A2} in the first quadrant at an angle α with the x-axis, and the second moves with speed v_{B2} in the fourth quadrant at an angle β with the x-axis (Fig. 8.13). (a) Write the equations expressing conservation of linear momentum in the x- and y-directions. (b) Square the equations from part (a) and add them. (c) Now introduce the fact that the collision is elastic. (d) Prove that $\alpha + \beta = \pi/2$. (You have shown that this equation is obeyed in any elastic, off-center collision between objects of equal mass when an object is initially at rest.)

Ex. 96 A 20.0 kg projectile is fired at an angle of 60.0° above the horizontal with a speed of 80.0 m/s. At the highest point of its trajectory, the projectile explodes into two fragments with equal mass, one of which falls vertically with zero initial speed. Ignore air resistance. (a) How far from the point of firing does the other fragment strike if the terrain is level? (b) How much energy is released during the explosion?

Delta V Nomogram v01

Lay straightedge between any two known values
and read the unknown value where edge crosses the scale.
dV = Ve.ln(R)
In the delta V scale:
Hohman are minimum-delta-V, maximum-duration impulse trajectories.
I-3 are impulse trajectories near the transition between delta-V levels
for high impulse trajectories and low brachistochrone trajectories.
(it is a hyperbolic solar escape orbit plus 30 km/s).
I-2 is an impulse trajectory in-between Hohman and I-3
(it is equivalent to an elliptical orbit from Mercury to Pluto)
Winchell Chung (Nyrath the Nearly Wise)
http://www.ProjectRho.com/rocket/index.html

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Winchell Chung (Nyrath the Nearly Wise)
http://www.ProjectRho.com/rocket/index.html
Exhaust Velocity (m/s)
Specific Impulse (sec)
Delta V (m/s)
Mass Ratio
Propellant Fraction
(no way) 25
(dV = 3xEv) 20
(close to limits on the possible) 10
(dV = 2xEv) 7.4
(limit for economical cargo ship) 4
(dV = Ev) 2.7
(dV = Ev/2) 1.64