Chapter 13
Gravitation

1 Newton's Law of Gravitation

Along with his three laws of motion, Isaac Newton also published his **law of gravitation** in 1687.

> **Every particle of matter in the universe attracts every other particle with a force that is directly proportional to the product of the masses of the particles and inversely proportional to the square of the distance between them.**

$$F_g = \frac{Gm_1m_2}{r^2}$$

where $G = 6.6742(10) \times 10^{-11}$ N·m^2/kg^2.

- The force between two objects are equal and opposite (Newton's 3$^{\text{rd}}$ Law)
- Gravitational forces combine vectorially. If two masses exert forces on a third, the *total* force on the third mass is the vector sum of the individual forces of the first two. This is the principle of *superposition*.
- Gravity is always attractive.

Ex. 1 What is the ratio of the gravitational pull of the sun on the moon to that of the earth on the moon? (Assume the distance of the moon from the sun can be approximated by the distance of the earth from the sun–1.50×10^{11}meters.) Use the data in Appendix F. Is it more accurate to say that the moon orbits the earth, or that the moon orbits the sun? $\mathrm{M_{earth}} = 5.97 \times 10^{24}$ kg, $\mathrm{m_{moon}} = 7.35 \times 10^{22}$ kg, $\mathrm{M_{sun}} = 1.99 \times 10^{30}$ kg, $\mathrm{d_{moon}} = 3.84 \times 10^{8}$ m.

Ex. 4 Two uniform spheres, each with mass M and radius R, touch one another. What is the magnitude of their gravitational force of attraction?

2 Weight

When calculating the gravitation pull of an object near the surface of the earth, the quantity $GM_{\text{earth}}/R_{\text{earth}}^2$ keeps appearing. It is more convenient to calculate this quantity and give it a name.

$$F_g = m\left(\frac{GM_{\text{earth}}}{R_{\text{earth}}^2}\right) = m\,g$$

where m is the mass of the object. The quantity mg is called "the weight" of the object, and $g = 9.8$ m/s^2, is the acceleration near the surface of the earth.

Ex. 14 Rhea, one of Saturn's moons, has a radius of 765 km and an acceleration due to gravity of 0.278 m/s^2 at its surface. Calculate its mass and average density.

Test your understanding: The mass of Mars is only 11% as large as the earth's mass. Why, then, isn't the surface gravity on Mars 11% as great as on earth?

3 Gravitational Potential Energy

Recall that our previous expression for gravitational potential energy near the surface of the earth is $U = mgy$. What is the gravitational potential energy as we move further away from the surface of the earth?

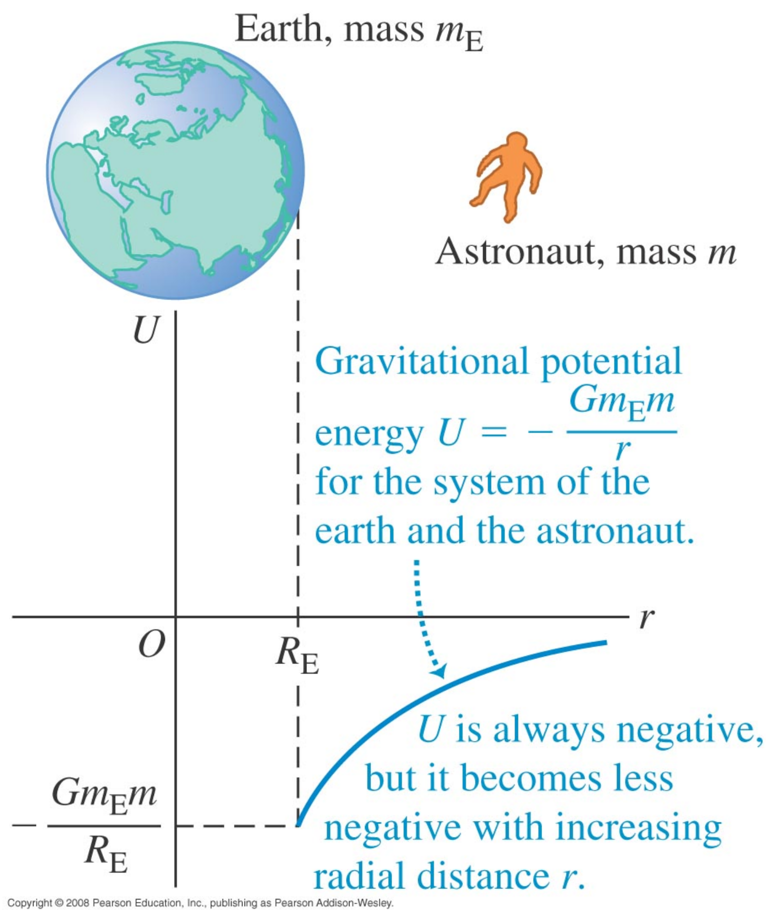
$$W_{\text{grav}} = \int_{r_1}^{r_2} F_r\,dr \qquad \text{where} \qquad F_r = -\frac{Gm_E m}{r^2}$$

$$W_{\text{grav}} = Gm_E m\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

where the gravitational potential energy is:

$$U_{\text{grav}} = -\frac{Gm_E m}{r}$$

N.B. Don't confuse the expressions for gravitational force with gravitational potential energy.



3.1 Escape Velocity

Use conservation of energy to find the escape velocity:

$$E_{\text{surface}} = E_{\infty}$$

$$K_{\text{surface}} + U_{\text{surface}} = K_{\infty} + U_{\infty}$$

$$\frac{1}{2}mv_{\text{escape}}^2 - \frac{GM_E\, m}{R_E} = 0 - \frac{GM_E\, m}{\infty}$$

$$v_{\text{escape}} = \sqrt{\frac{2GM_E}{R_E}}$$

The escape velocity does not depend on the mass m of the projectile.

Ex. 19 A planet orbiting a distant star has radius 3.24×10^6 m. The escape speed for an object lauched from this planet's surface is 7.65×10^3 m/s. What is the acceleration due to gravity at the surface of the planet?

4 The Motion of Satellites

By using Newton's 2nd Law and Newton's Law of Universal Gravitation, we can calculate the following:

$$v = \sqrt{\frac{G\,M_E}{r}}$$

and

$$T = \frac{2\pi r}{v} = \frac{2\pi r^{3/2}}{\sqrt{G\,M_E}}$$

and

$$E = K + U = -\frac{G\,M_E\,m}{2r} \qquad \text{(circular orbit)}$$

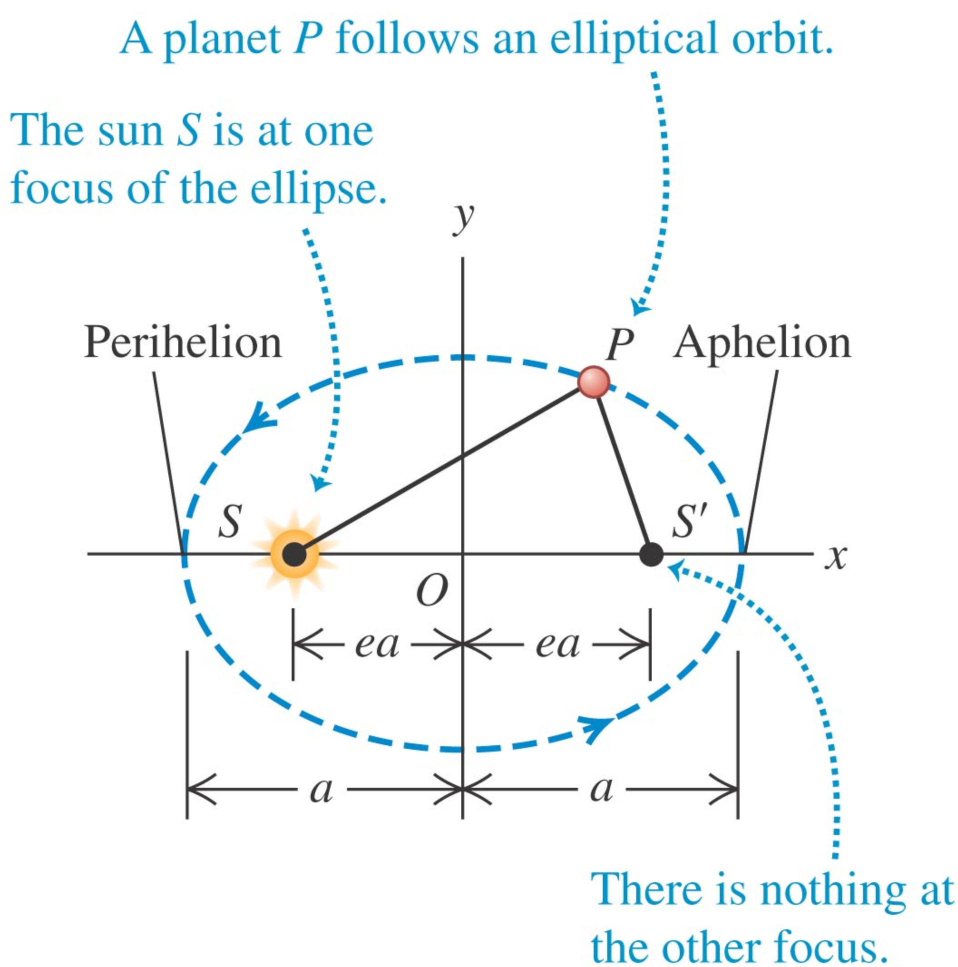
Ex. 25 For a satellite to be in a circular orbit 890 km above the surface of the earth, (a) what orbital speed must it be given and (b) what is the period of the orbit (in hours)? $\mathrm{R_{earth}} = 6.37\times10^6$ m

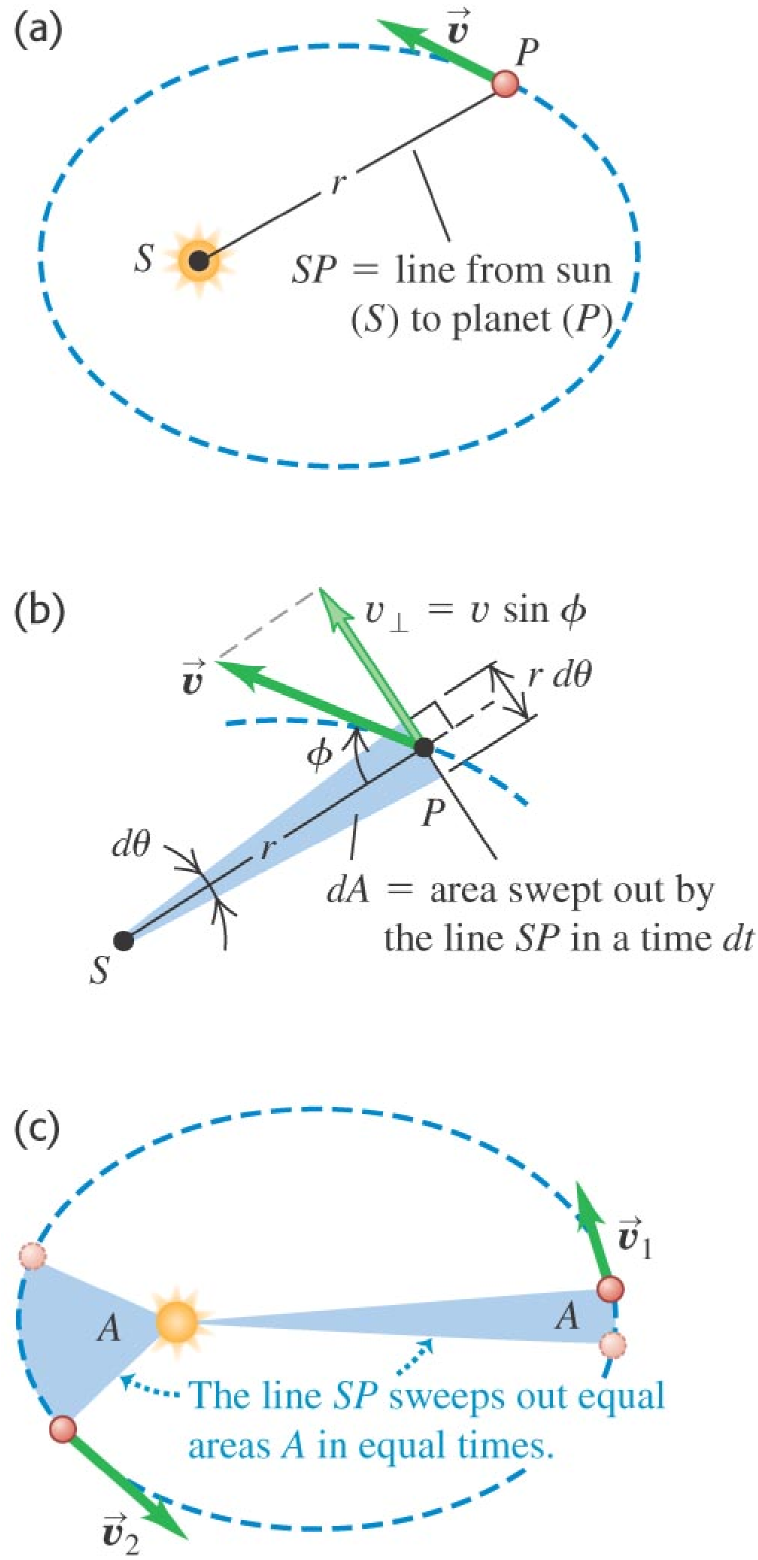
5 Kepler's Laws and the Motion of Planets

1^{st} **Law** Each planet moves in an elliptical orbit, with the sun at one focus of the ellipse.

2^{nd} **Law** A line from the sun to a given planet sweeps out equal areas in equal times.

3^{rd} **Law** the periods of the planets are proportional to the $\frac{3}{2}$ powers of the major axis lengths of their orbits.




(a)
$\vec{v}$
P
r
S
SP = line from sun (S) to planet (P)
(b)
$v_\perp = v \sin\phi$
$\vec{v}$
$r\,d\theta$
ϕ
P
$d\theta$
r
S
dA = area swept out by the line SP in a time dt
(c)
$\vec{v}_1$
A
A
The line SP sweeps out equal areas A in equal times.
$\vec{v}_2$

Ex. 30 **Hot Jupiters.** In 2004 astronomers reported the discovery of a large Jupiter-sized planet orbiting very close to the star HD 179949 (hence the term "hot Jupiter"). The orbit was just $\frac{1}{9}$ the distance of Mercury from our sun, and it takes the planet only 3.09 days to make one orbit (assumed to be circular). (a) What is the mass of the star? Express your answer in kilograms and as a multiple of our sun's mass. (b) How fast (in km/s) is this planet moving? $d_{\text{Mercury}} = 5.79 \times 10^{10}$ m

Graviational potential energy

1. Potential energy for a mass at a single point.

$$F = -\frac{dU}{dr} \qquad dU = -F\,dr \qquad \int_\infty^r dU = -\int_\infty^r F\,dr = -\int_\infty^r \frac{-GMm}{r^2}\,dr$$

$$U(r) - U(\infty) \;=\; U(r) \;=\; -\frac{GMm}{r}$$

2. The change in potential energy between two points.

$$\Delta U_{\rm grav} = U_{\rm final} - U_{\rm initial} \;=\; -\frac{GMm}{r_{\rm f}} \;-\; \left(-\frac{GMm}{r_{\rm i}}\right)$$

Work-Energy Theorem $\qquad W = \Delta KE$

However, since gravity is a conservative force (we have a PE function, $U_{\rm grav}$)

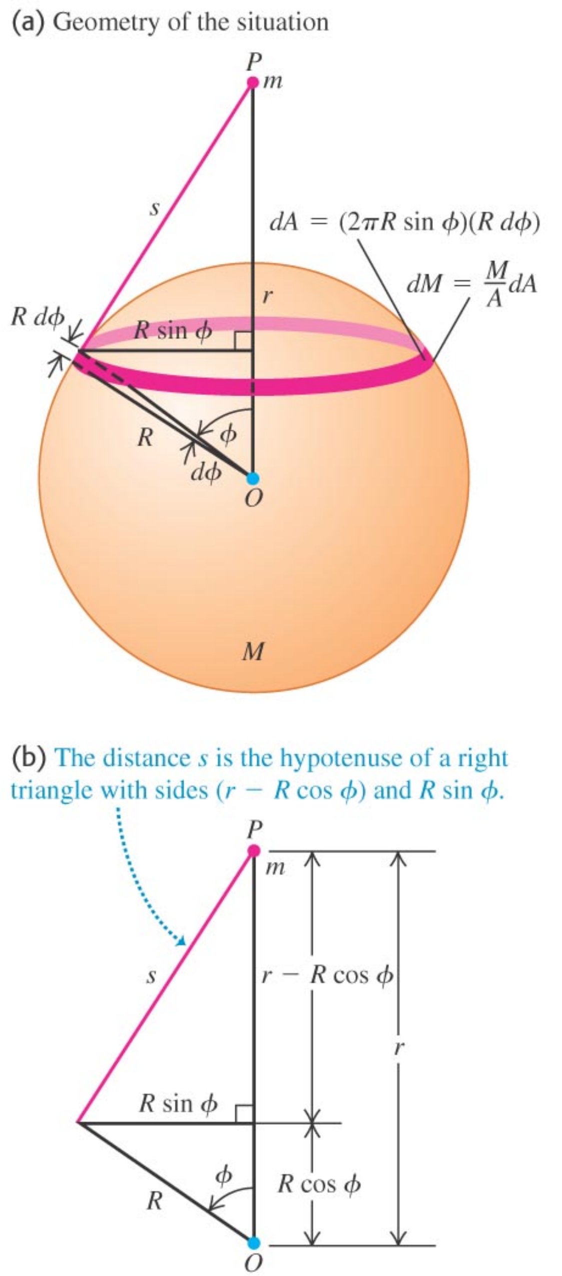
$$W_{\rm grav} = -\,\Delta U_{\rm grav} \;=\; GMm\left(\frac{1}{r_{\rm f}} - \frac{1}{r_{\rm i}}\right) \;=\; \frac{1}{2}mv_{\rm f}^2 - \frac{1}{2}mv_{\rm i}^2$$

6 Spherical Mass Distributions

The Two Shell Theorem

Shell Theorem #1 *A uniformly dense spherical shell attracts an external particle as if all the mass of the shell were concentrated at its center.*

Shell Theorem #2 *A uniformly dense spherical shell exerts no gravitational force on a particle located anywhere inside it.*



Some key steps in the derivation

1. the density of a thin spherical shell $\rho = M/V = M/4\pi R^2\, t$

2. the force on m due to small mass (dm) on the sphere

$$dF = -G\,\frac{m\,dm}{s^2}\,(\cos\alpha) \qquad \text{where} \qquad \cos\alpha = \frac{r - R\cos\phi}{s}$$

where α is the angle between line segments s and r at point P.

3. the force on m due to an annular ring of mass on the sphere

$$dF = -G\,\frac{m\,dm_A}{s^2}\left(\frac{r - R\cos\phi}{s}\right)$$

where dm_A is the mass in the annular ring

$$dm_A = \rho\,dV = \rho\,[2\pi(R\,\sin\phi)\,(R\,d\phi)\,t]$$

where t is the thickness of the *thin* spherical shell, where $t \ll R$.

4. the relationship between ϕ and α is correct for either *inside* or *outside* the spherical shell:

$$\cos\alpha \;=\; \frac{r - R\,\cos\phi}{s}$$

5. using the law of cosines

$$s^2 \;=\; r^2 + R^2 - 2rR\,\cos\phi$$

$$2s\,ds \;=\; 2rR\,\sin\phi\,d\phi$$

6. making s the only independent variable

$$\sin\phi\;d\phi \;=\; \frac{s}{rR}\,ds$$

$$dF = -\frac{\pi G t\rho m R}{r^2}\left(\frac{r^2 - R^2}{s^2} + 1\right)\,ds$$

7. **Outside the spherical shell,** the limits of s are from $r - R$ to $r + R$.

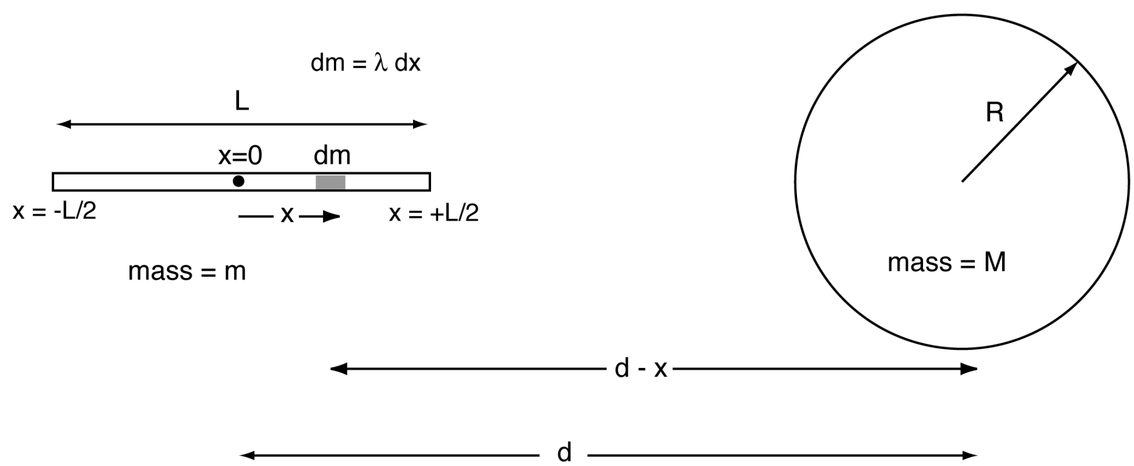
$$F = \int dF = -\frac{\pi G t \rho m R}{r^2} \int_{r-R}^{r+R} \left(\frac{r^2 - R^2}{s^2} + 1 \right) ds = -\frac{\pi G t \rho m R}{r^2} (4R) = -\frac{GMm}{r^2}$$

8. **Inside the spherical shell,** the limits of s are from $R - r$ to $R + r$.

$$F = \int dF = -\frac{\pi G t \rho m R}{r^2} \int_{R-r}^{R+r} \left(\frac{r^2 - R^2}{s^2} + 1 \right) ds = -\frac{\pi G t \rho m R}{r^2} (0) = 0$$

Example

Suppose we have a *uniform* solid sphere of mass M and a *thin* cylindrical bar of mass m separated by a distance d between their respective centers of mass.



$$dF = \frac{GM\,\lambda\,dx}{(d-x)^2} \qquad F = GM\lambda \int_{-L/2}^{+L/2} \frac{dx}{(d-x)^2} = \frac{GMm}{d^2} \frac{1}{\left(1 - \frac{L^2}{4d^2}\right)} \qquad (1)$$

where dF is the force on dm, and $dF = GM\,dm/(d-x)^2$, $dm = \lambda dx$, and $m = \lambda L$.

The total force on the rod (F) of length L is **not** equivalent to taking the mass of the rod and concentrating it at its center-of-mass.

Problem What is the total force on the rod if it is rotated 90^o about its center-of-mass?

$$dF_x = dF \cos\alpha = \frac{GM\lambda\, dy}{(d^2+y^2)} \cos\alpha \; = \; \frac{GM\lambda\; d\; dy}{(d^2+y^2)^{3/2}}$$

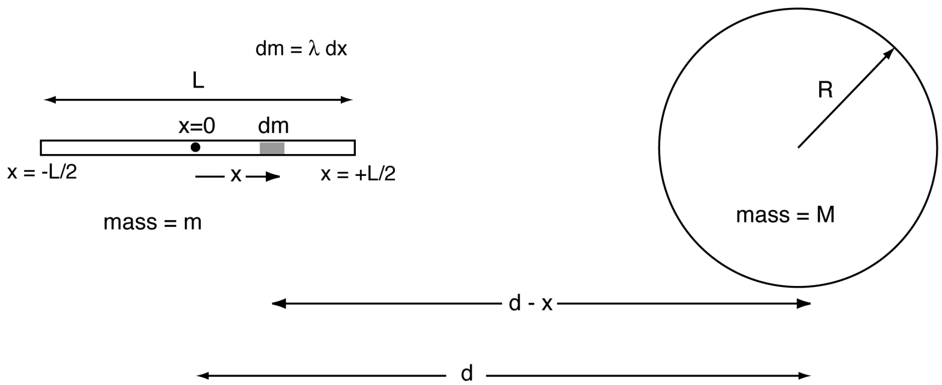
$$F = GM\lambda\, d \int_{-L/2}^{L/2} \frac{dy}{(d^2+y^2)^{3/2}} \; = \; \frac{GMm}{d^2} \frac{1}{\sqrt{1+\frac{L^2}{4d^2}}} \tag{2}$$

Again, the total force on the rod of length L is **not** equivalent to taking the mass of the rod and concentrating it at its center-of-mass.

The *sphere* appears to be the only “simple” object where one can calculate the force by concentrating all its mass at the center-of-mass.

7 Gravitational Gradient Alignment

Let's go back to our problem where we have a thin, cylindrical bar of mass m and a uniform sphere of mass M separated by a distance d. What is the potential energy of this two-mass system?



$$dU = -\frac{GM\,\lambda\,dx}{(d-x)}$$

$$U_{\text{horizontal}} = -GM\lambda \int_{-L/2}^{+L/2} \frac{dx}{(d-x)} \;=\; -\frac{GMm}{L}\,\ell n\left(\frac{1+\frac{L}{2d}}{1-\frac{L}{2d}}\right) \tag{3}$$

where dU is the potential energy between dm and M, and $dU \;= -\,GM\,dm/(d-x)$, and $dm = \lambda dx$, and $m = \lambda L$.

The total potential energy (U) between the rod of length L and the sphere of mass M is **not** equivalent to taking the mass of the rod and concentrating it at its center-of-mass. That is, $U \neq -GMm/d$

Problem What is the total potential energy if the rod is rotated 90^o about its center-of-mass?

$$dU = -\frac{GM\ dm}{\sqrt{d^2+y^2}} \;=\; -\frac{GM\ \lambda\ dy}{(d^2+y^2)^{1/2}}$$

[

$$U_{\text{vertical}} = -GM\lambda \int_{-L/2}^{L/2} \frac{dy}{(d^2+y^2)^{1/2}} \;=\; -\frac{GMm}{L}\;\left(2\;\sinh^{-1}\left(\frac{L}{2d}\right)\right) \tag{4}$$

Examining the two results (U_{vertical} and $U_{\text{horizontal}}$), we can see that

$$U_{\text{horizontal}} \;<\; U_{\text{vertical}} \tag{5}$$

for all values of $L/2d$ where $d > L/2$. The rod prefers to be in the *horizontal* orientation compared to the *vertical* orientation. **Investigating this in more detail we find the following.**

$$U_{\text{horizontal}} = -\;\frac{GMm}{L}\;\ell n\left(\frac{1+\frac{L}{2d}}{1-\frac{L}{2d}}\right) \;\simeq\; -\frac{GMm}{L}\;\ell n\left(1+\frac{L}{2}+\cdots\right)$$

Using the approximation that $\ell n\left(\frac{1+x}{1-x}\right) \simeq 2x + \frac{2x^3}{3} + \cdots$, we have that:

$$U_{\text{horizontal}} \simeq -\frac{GMm}{L}\left(2\frac{L}{2d}+\frac{2}{3}\frac{L^3}{8d^3}+\cdots\right) \simeq -\;\frac{GMm}{d}\left(1+\frac{1}{12}\frac{L^2}{d^2}+\cdots\right) \tag{6}$$

While in the vertical orientation, we have:

$$U_{\text{vertical}} = \;-\frac{GMm}{L}\;\left(2\;\sinh^{-1}\left(\frac{L}{2d}\right)\right) \;\simeq\; -\frac{GMm}{L}\left(2x-\frac{x^3}{3}+\frac{3x^5}{20}+O\left(x^6\right)\right)$$

where $x = \left(\frac{L}{2d}\right)$. Our final result for the potential energy in the *vertical* orientation is:

$$U_{\text{vertical}} \simeq -\frac{GMm}{L}\left(\frac{L}{d}-\frac{1}{3}\frac{L^3}{8d^3}+\cdots\right) \;\simeq\; -\frac{GMm}{d}\left(1-\frac{L^2}{24d^2}+\cdots\right) \tag{7}$$

Comparing equations 6 and 7, we see that the *horizontal* orientation has a slightly lower potential energy when compared to the *vertical* orientation.

Another application of the shell theorem

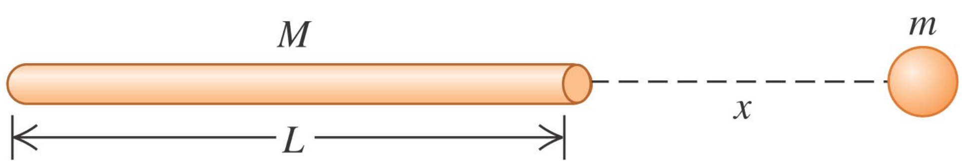
Suppose a hole is drilled through the earth M_E passing through its center. What is the force on a mass m as a function of r, its distance from the center of the earth?

$$F(r) = -\frac{GM_E\, m}{R_E^3}\, r \quad = \quad -kr \qquad \text{(Hooke's Law)}$$

The mass m undergoes *simple harmonic motion* similar to a mass on a spring. We'll study this later in chapter 17.

How does g vary with r, the distance from the center of the earth?

Ex. 38 A thin, uniform rod has length L and mass M. A small uniform sphere of mass m is placed a distance x from one end of the rod, along the axis of the rod (Fig. 13.38). (a) Calculate the gravitational potential energy of the rod-sphere system. Take the potential energy to be zero when rod and sphere are infinitely far apart. Show that your answer reduces to the expected results when x is much larger than L (*Hint:* Use the power series expansion for $\ell n(1+x)$ given in Appendix B.) (b) Use $F_x = -dU/dx$ to find the magnitude and direction of the gravitational force exerted on the sphere by the rod (see Section 7.4). Show that your answer reduces to the expected results when x is much larger than L



We have solved this problem "up above" with slightly different definitions of the variables.

$$dU = -\frac{G\, dM\, m}{r} = -\frac{G\,(\lambda\, ds)\, m}{x - s}$$

$$U = -Gm\lambda \int_{-L}^{0} \frac{ds}{x-s} = Gm\lambda\, \ell n \left(1 + \frac{L}{x}\right) \simeq -\frac{GmM}{x}\left(1 - \frac{L}{2x} + \cdots\right)$$

using the approximation that $\ell n(1+x) \simeq x - x^2/2 + x^3/3 - \cdots$.

8 Apparent Weight and the Earth's Rotation

9 Black Holes

Black holes are massive objects due the collapse of massive stars after a supernova. If an object approaches a black hole and its distance is less than a critical distance (called the Schwarzchild radius), no amount of kinetic energy can propel it away from the black hole. The Schwarzchild radius is sometimes called the *event horizon*.

$$R_S = \frac{2GM}{c^2} \qquad \text{(Schwarzchild radius)}$$

How can you determine the mass of a black hole? You can measure the *period* and the *semi-major axis* of a satellite orbiting the black hole.

$$T = \frac{2\pi}{\sqrt{GM_{\mathrm{bh}}}}\, a^{3/2}$$

Ex. 43 **At the Galaxy's Core.** Astronomers have observed a small, massive object at the center of our Milky Way galaxy (see Section 13.8). A ring of material orbits this massive object; the ring has a diameter of about 15 light-years and an orbital speed of about 200 km/s. (a) Determine the mass of the object at the center of the Milky Way galaxy. Give your answer both in kilograms and in solar masses (one solar mass is the mass of the sun). (b) Observations of stars, as well as theories of the structure of stars, suggest that it is impossible for a single star to have a mass of more than about 50 solar masses. Can this massive object be a single, ordinary star? (c) Many astronomers believe that the massive object at the center of the Milky Way galaxy is a black hole. If so, what must the Schwarzschild radius of this black hole be? Would a black hole of this size fit inside the earth's orbit around the sun?

Prob. 56 Your starship, the *Aimless Wanderer*, lands on the mysterious planet Mongo. As chief scientist-engineer, you make the following measurements: A 2.50-kg stone thrown upward from the ground at 12.0 m/s returns to the ground in 4.80 s; the circumference of Mongo at the equator is 2.00×10^5 km; and there is no appreciable atmosphere on Mongo. The starship commander, Captain Confusion, asks for the following information: a) What is the mass of Mongo? b) If the *Aimless Wanderer* goes into a circular orbit 30,000 km above the surface of Mongo, how many hours will it take the ship to complete one orbit.

9.1 Perigee and Apogee About Different Astronomical Objects

Here is an interesting web page that documents all the different names used for specific *apsides* of astronomical satellites orbiting a more massive object.

Prob. 79 **Interplanetary Navigation:** The most efficient way to send a spacecraft from the earth to another planet is by using a *Hohmann transfer orbit* (**Fig. P13.79**). If the orbits of the departure and destination planets are circular, the Hohmann transfer orbit is an elliptical orbit whose perihelion and aphelion are tangent to the orbits of the two planets. The rockets are fired briefly at the departure planet to put the spacecraft into the transfer orbit; the spacecraft then coasts until it reaches the destination planet. The rockets are then fired again to put the spacecraft into the same orbit about the sun as the destination planet. a) For a flight from the earth to Mars: in what direction must the rockets be fired at the earth and at Mars–in the direction of motion, or opposite the direction of motion? What about from a flight from Mars to the earth? b) How long does a one-way trip from the earth to Mars take, between the firings of the rockets? c) To reach Mars from the earth, the launch must be timed so that Mars will be at the right spot when the spacecraft reaches Mars's orbit around the sun. At launch, what must the angle between a sun-Mars line and a sun-earth line be? Use data from Appendix F.

$d_{\text{mars}} = 2.28 \times 10^{11}$ m, $T_{\text{mars}} = 687$ days, and $d_{\text{earth}} = 1.50 \times 10^{11}$ m.

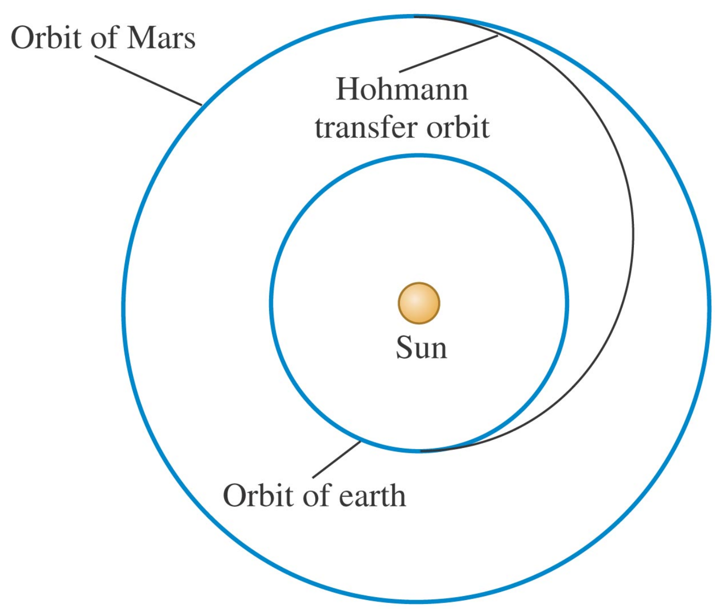


Figure 1: Figure P13.79 from University Physics 15th edition.

T/2 = 259.3 days. $\theta = 44.1^{o}$.